E. Dummit's Math 1465 $\sim$ Intensive Mathematical Reasoning, Fall 2024 $\sim$ Homework 2, due Tue Sep 17th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Suppose $A = \{1,3,9\}$, $B = \{6,7,8,9\}$, and $C = \{2,3,5,7\}$, with universal set $U = \{1,2,3,4,5,6,7,8,9\}$. Calculate the following:

   (a) $A \cap B$
   (b) $A \cup C$
   (c) $(A \cup B) \cup C$
   (d) $A \cup (B \cup C)$
   (e) $A \cap (B \cup C)$
   (f) $(A \cap B) \cup C$
   (g) $B^c \cap C^c$
   (h) $(A^c \cup B) \cap (B^c \cup C)$
   (i) $A \times B$
   (j) $B \times A$
   (k) $(A \times B) \cap (B \times A)$
   (l) $(A \cap B) \times (B \cup C)$

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2. Suppose $A = \{1, 2, \{1\}, \emptyset\}$. Identify each of the statements below as true or as false, and give a brief (1-sentence) explanation why.

   (a) $\emptyset \in A$.
   (b) $\emptyset \subseteq A$.
   (c) $\{1\} \in A$.
   (d) $\{1\} \subseteq A$.
   (e) $\{1, 2\} \in A$.
   (f) $\{1, 2\} \subseteq A$.
   (g) $\{\{1\}\} \in A$.
   (h) $\{\{1\}\} \subseteq A$.

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3. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

   (a) Proposition: If A and B are sets with $A \cup B = \{1,2,3\}$ then $A \neq \{1\}$ and $B \neq \{2\}$.
   Proof: Suppose by way of contradiction that the desired result is false. Then $A = \{1\}$ and $B = \{2\}$, but then $A \cup B = \{1, 2\}$ which contradicts the assumption that $A \cup B = \{1, 2, 3\}$. This is impossible, so we must have $A \neq \{1\}$ and $B \neq \{2\}$.

   (b) Proposition: For any sets A and B inside a universal set U, $(A \cup B) \cap B^c = A$.
   Proof: The union $A \cup B$ consists of all elements in A or in B. Taking the intersection with B^c removes all elements that are in B, leaving only the elements in A. Therefore, $(A \cup B) \cap B^c = A$.

   (c) Proposition: For any sets A and B inside a universal set U, if $A^c \subseteq B^c$ then $B \subseteq A$.
   Proof: Suppose B is a subset of A, meaning that if $x \in B$ then $x \in A$. Taking the contrapositive of this statement shows that if $x \notin A$ then $x \notin B$, which is the same as saying that if $x \in A^c$ then $x \in B^c$. This shows $A^c \subseteq B^c$, as required.

   (d) Proposition: If A, B, C are sets with $A \subseteq C$, $B \subseteq C$, and $x \in A$, then $x \in B$.
   Proof: Suppose by way of contradiction that $x \notin B$. Since $x \in A$ and $A \subseteq C$, this means $x \in C$. Also, since $x \notin B$ and $B \subseteq C$, this means $x \notin C$. These two statements contradict each other, which is impossible. Therefore, we must have $x \in B$.

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4. Wikipedia has many articles. Some of these articles are lists, such as the article "List of American mathematicians". There are even articles which list other lists, such as the article "List of lists of mathematical topics". Some of these lists of lists contain themselves, such as the article "List of lists of lists".

   (a) Consider the Wikipedia article titled "List of lists that do not contain themselves". Can this article be listed on itself? (Assume, as most students do, that everything on Wikipedia is accurate.)

   (b) The only true test of a hypothesis is empirical evidence. To this end, what *actually happens* when you visit the Wikipedia article "List of lists that do not contain themselves"?

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5. In addition to union and intersection, there are a few other set operations that arise from time to time. Two of these are the set difference $A\backslash B = \{x \in A : x \notin B\}$, the set of elements of A not in B, and the symmetric difference $A\Delta B = (A\backslash B) \cup (B\backslash A)$, the elements in either A or B but not both. (Observe that $A\Delta B = B\Delta A$, whence the name *symmetric* difference.)

   (a) If $A = \{1,2,3,4,5\}$, $B = \{2,4,6\}$, and $C = \{1,3,5\}$, find $A\backslash B$, $B\backslash A$, $A\backslash C$, $C\backslash A$, $B\backslash C$, and $C\backslash B$.

   (b) If $A = \{1,2,3,4,5\}$, $B = \{2,4,6\}$, and $C = \{1,3,5\}$, find $A\Delta B$, $A\Delta C$, and $B\Delta C$.

   (c) For each of the following statements, decide whether it is true or whether it is false using a Venn diagram or otherwise:

      i. $A\backslash B = A \cap B^c$
      ii. $A\Delta B = (A \cap B^c) \cup (A^c \cap B)$
      iii. $(A\backslash B) \cup B = A$
      iv. $(A\backslash B) \cup (A\backslash B^c) = A$
      v. $A\Delta B = (A \cup B)\backslash(A \cap B)$
      vi. $(A\backslash B) \cap (B\backslash A) = \emptyset$
      vii. $(A\Delta B)^c = A^c\backslash B$
      viii. $(A\backslash B)\backslash C = (A\backslash C)\backslash B$
      ix. $(A\Delta B)\Delta C = A\Delta(B\Delta C)$
      x. $(A \cap B)\Delta C = (A\Delta C)\Delta(A\backslash B)$

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. A common strategy to show that two sets S and T are equal is to show that $S \subseteq T$ (by establishing that $x \in S$ implies $x \in T$) and also that $T \subseteq S$ (by establishing that $x \in T$ implies $x \in S$). Using this method, or otherwise, and supposing that A, B, and C are any sets contained in a universal set U, provide a rigorous proof of each of the following statements (in particular, you may *not* appeal to Venn diagrams in your solutions and must use *only* formal properties of set containments):

   (a) Prove that $A \cap (A \cap B) = A \cap B$.

   (b) Prove that $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

   (c) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

   Remark: Note that some of these parts ask for a proof of a result stated but not explicitly proven in the course notes. As such, you may NOT quote those results in the responses (e.g., your response to part (c) cannot be "This follows immediately from the distributive laws for Cartesian products"), because it would be circular logic.

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7. Suppose A and B are sets. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

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8. In some mathematical arguments, there can arise several different possible cases. In such situations, it can be useful to break the argument apart and analyze each of the possible cases separately, rather than trying to deal with them all at once. Indeed, the idea of breaking into cases is the procedure underlying the use of truth tables: we simply evaluate logical expressions in all possible cases to see whether they agree. Using this idea, or otherwise, prove the following:

   (a) If A, B, C are any sets, prove that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

   (b) If A, B, C are any sets, prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, and deduce in fact that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (c) If A, B are any subsets of a universal set U, prove that $(A \cup B)^c = A^c \cap B^c$. [Hint: For any $x \in U$, there are four possible cases: either $x \in A, x \in B$ or $x \in A, x \notin B$ or $x \notin A, x \in B$ or $x \notin A, x \notin B$.]
