1. Match the erroneous proofs (a)-(e) with the reasons (1)-(5) they are not valid proofs of the claims.

   (a) Proposition: The integer 2 is odd.
   Proof: It is a fact about odd integers that any odd integer plus any odd integer always gives an even integer. Because $2 + 2 = 4$, and 4 is an even integer, this means 2 must be odd.

   - The proposition is obviously false. Although the fact about odd integers is true, and 4 is indeed even, it is also true that an even integer plus an even integer is also even.
   - The information stated is (indeed) consistent with saying that 2 is odd, but it is also consistent with saying that 2 is even. The proof is wrong because it references a true statement but this statement does not imply the conclusion, which is reason $\boxed{(5)}$.

   (b) Proposition: For any proposition P, $\neg(\neg(\neg P)))$ is always false.
   Proof: If P is true, then $\neg(\neg(\neg P)))$ is false, and if P is false then $\neg(\neg(\neg P)))$ is also false.

   - The first case is correct, as $\neg(\neg(\neg T))) = \neg(\neg F) = \neg T = F$ is false, but the second case is wrong since $\neg(\neg(\neg F))) = T$ is true. The proof is wrong because it is a list of examples where not all the examples are correct, which is reason $\boxed{(4)}$.

   (c) Proposition: If x is an integer and $3x - 2 = 7$, then $x = 3$.
   Proof: Suppose $x = 3$. Then $3x - 2 = 3(3) - 2 = 7$. Therefore, if $3x - 2 = 7$, then $x = 3$.

   - The proposition is true. However, because the proof begins by assuming $x = 3$ and concluding that $3x - 2 = 7$, it is wrong for reason $\boxed{(1)}$: the proof actually shows the converse of the required statement.
   - The correct proof would start by assuming that $3x - 2 = 7$ and deduce that $x = 3$.

   (d) Proposition: Every odd number greater than 1, except for 9, is prime.
   Proof: Clearly, 3 is prime, 5 is prime, 7 is prime, 9 is not prime, 11 is prime, and 13 is prime. Since we have excluded 9, all odd numbers greater than 1 are prime.

   - The "proof" merely lists some examples, and does not give any actual argument, so it is not actually a proof at all. The proof is a list of examples only showing the result in some cases, not all cases, which is reason $\boxed{(2)}$.

   (e) Proposition: For an integer m, m is even if and only if m^2 is even.
   Proof: Suppose m is even. Then $m = 2k$ for some integer k, meaning that $m^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$ and thus m^2 is even. Therefore, m is even if and only if m^2 is even.

   - The proposition is actually true, but is an "if and only if" statement, while the proof only establishes one direction (namely, if m is even then m^2 is even).
   - The error is that the proof is missing an argument showing "if m^2 is even, then m is even": the proof only shows one direction of a biconditional, which is reason $\boxed{(3)}$.

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2. Find a counterexample to each of the following statements (make sure to explain briefly why your example is a counterexample):

   (a) If n is any integer, then n^2 is greater than n.

   - Since $0^2 = 0$ and $1^2 = 1$, both $n = \boxed{0}$ and $n = \boxed{1}$ are counterexamples.

   (b) If a and b are positive integers, then $ab + a + b \neq 7$.

   - We want to find a, b with $ab+a+b = 7$. Trying some small choices for a and b show that $(a, b) = (1, 3)$ or $(3, 1)$ works, so these are both counterexamples.

   (c) If P and Q are any propositions, then $(\neg P \wedge \neg Q) \vee (P \wedge Q)$ is always true.

   - Trying various truth values we can see that if one of P, Q is true and the other is false then $(\neg P \wedge \neg Q)$ and $P \wedge Q$ are both false, so $(\neg P \wedge \neg Q) \vee (P \wedge Q)$ is also false.

(d) If P and Q are any propositions, then $\neg P \wedge Q$ is logically equivalent to $\neg(P \vee Q)$.

- If P is false and Q is true, then $\neg P \wedge Q$ is true while $\neg(P \vee Q)$ is false.

(e) There is no integer n such that both n and $3n+1$ are prime numbers.

- For $n = 2$, both $n = 2$ and $3n + 1 = 7$ are prime, so $n = \boxed{2}$ is a counterexample.

(f) If p is a prime number, then $2^p - 1$ is also a prime number.

- We try testing various primes to see if $2^p - 1$ is prime.
- We can see $2^2 - 1 = 3$ is prime, $2^3 - 1 = 7$ is prime, $2^5 - 1 = 31$ is prime, $2^7 - 1 = 127$ is prime, but $2^{11} - 1 = 2047 = 23 \cdot 89$ is not prime.
- Therefore, $p = \boxed{11}$ is a counterexample to the given statement.

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3. Write explicitly the converse, inverse, and contrapositive of the following conditional statements:

(a) If you do not study for your exams, then you will get bad grades.

- The converse is: "If you get bad grades, then you did not study for your exams".
- The inverse is: "If you do study for your exams, then you will not get bad grades".
- The contrapositive is: "If you did not get bad grades, then you did study for your exams".

(b) If n is an odd integer greater than 7, then n is the sum of three odd primes.

- The converse is: "If n is the sum of three odd primes, then n is an odd integer greater than 7".
- The inverse is: "If n is not an odd integer greater than 7, then n is not the sum of three odd primes".
- The contrapositive is: "If n is not the sum of three odd primes, then n is not an odd integer greater than 7".

(c) If you want to bake a cake, then you must have eggs and flour.

- The converse is: "If you have eggs and flour, then you want to bake a cake".
- The inverse is: "If you do not want to bake a cake, then you must not have eggs and flour".
- The contrapositive is: "If you do not have eggs and flour, then you do not want to bake a cake".

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4. Answer the following questions based on the "Tips on Problem Solving" course page:
https://dummit.cos.northeastern.edu/teaching_fa24_1465/1465_problemsolving.html

(a) What are five different general tips on problem solving presented at the link?

- Nine tips presented are (1) Try coming up with an example, and work out the result in that special case, (2) Try playing around with the hypotheses, (3) Try to establish a simpler result, (4) Draw a picture, (5) Don't try to reinvent the wheel if you don't have to, (6) See if you can apply a big theorem (or a little one), (7) Ask dumb questions, (8) Talk to other people, (9) Be patient.

(b) In Math 1465, what can you do if you are having trouble getting started on a problem?

- If you are having trouble getting started on a problem, you can always start by writing down the hypotheses and (separately) the conclusion.

(c) Can you solve a problem that asks you to prove something by listing examples and explaining them?

- No: listing examples, even if they are explained, is not a solution to a problem that asks you to prove something. Most problems require you to prove that some result is true in every possible situation: just giving one or two examples where the result is true does not accomplish that task.

(d) How can you establish that two statements are equivalent (A if and only if B)?

- If you are trying to establish that two statements are equivalent (A if and only if B), try splitting it into two conditionals: if A then B, and if B then A. Each conditional gives you something to start with along with a goal you want to reach.

(e) How can you show that two numerical expressions are equal?

- When trying to show that two numerical expressions are equal, try starting with one side of the expression and doing transformations on it until you obtain the quantity on the other side.

(f) How can you show that an object is unique?

- If you are trying to show that an object is unique, suppose that there are two of them and then prove that they must be the same.

(g) In Math 1465, how can you tell whether your proof has any gaps, or needs more detail?

- Proofs should contain no gaps. Every statement in a proof should follow logically, in an obvious way, from previous statements or known facts. The definition of "in an obvious way" is a little bit subjective, but as a general rule, if your argument would not completely convince another student in this course who hasn't seen that particular problem, it needs more detail.

(h) Should your proofs be written in full sentences? Easy to read? Clear and concise?

- Standards for good writing also apply to proof-writing: you should use correct grammar, write in full sentences, organize sentences into paragraphs, etc. If the proof is hard to read (e.g., with lines erased or scribbled out, or missing parts added in separate areas of the page), you should rewrite it cleanly.
- Clarity and conciseness are important. A short and crisp proof is often much easier to understand than a longer one.

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5. Suppose that P, Q, and R are any propositions.

(a) Prove that if P and $P \Rightarrow Q$ are both true, then Q is also true. [Hint: Use a truth table to identify all cases in which both P and $P \Rightarrow Q$ are true.]

- We can make a truth table to see when P and $P \Rightarrow Q$ are both true. Then for each case where the answer is "yes", we can check whether Q is also true.
- This yields the following table:

P	Q	$P \Rightarrow Q$	Are P and $P \Rightarrow Q$ both true?	Is Q true?
T	T	T	Yes	Yes
T	F	T	No	–
F	T	F	No	–
F	F	F	No	–

- We can see that there is only one case where P and $P \Rightarrow Q$ are both true, and in that case the statement Q is also true. This establishes the result.

(b) Suppose that the statements "If it is raining, then it is cloudy" and "It is raining" are both true. Is the statement "It is cloudy" necessarily true? Explain. [Hint: Use (a).]

- Yes, it is true: let P be the proposition "It is raining" and Q be the proposition "It is cloudy".
- Then we are given that P and $P \Rightarrow Q$ are both true. By part (a), this means the statement Q is also true, meaning that it must be cloudy.

(c) Prove that if $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is also true.

- We can make a truth table to see when $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true. Then for each case where the answer is "yes", we can check whether $P \Rightarrow R$ is also true.
- This yields the following table:

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	Are $P \Rightarrow Q$ and $Q \Rightarrow R$ both true?	Is $P \Rightarrow R$ true?
T	T	T	T	T	Yes	Yes
T	T	F	T	F	No	–
T	F	T	F	T	No	–
T	F	F	F	T	No	–
F	T	T	T	T	Yes	Yes
F	T	F	T	F	No	–
F	F	T	T	T	Yes	Yes
F	F	F	T	T	Yes	Yes

- We can see in each of the four cases in which $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, the statement $P \Rightarrow R$ is also true. This establishes the result.

(d) Suppose that the statements "If it is raining, then it is cloudy" and "If it is cloudy, then people want to stay home" are both true. Is the statement "If it is raining, then people want to stay home" necessarily true? Explain. [Hint: Use (c).]

- Yes, it is true: let P be the proposition "It is raining", Q be the proposition "It is cloudy", and R be the statement "People want to stay home".
- Then we are given that $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true. By part (c), this means the statement $P \Rightarrow R$ is also true, meaning that if it is raining, then people want to stay home.

Remarks: The results of (b) and (d) here are forms of reasoning we often apply in deductive proofs (formally, (b) is known as *modus ponens* and (d) is known as *transitivity*). The point of this problem is to verify that these forms of deduction are in fact logically correct.

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6. Using a truth table or otherwise, determine whether each of following pairs of statements are equivalent. For those that are false, give an explicit counterexample (i.e., truth values for the propositions showing the statements are different):

(a) $A \wedge (A \vee B)$ and A.

- We form a truth table to see they are in fact equivalent. (In fact, this statement is one of the absorption laws.)

A	B	$A \vee B$	$A \wedge (A \vee B)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(b) $\neg(A \vee \neg B) \Rightarrow \neg B$ and $B \Rightarrow A$.

- We form a truth table to see they are in fact equivalent.

A	B	$\neg(A \vee \neg B)$	$\neg(A \vee \neg B) \Rightarrow \neg B$	$B \Rightarrow A$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	F
F	F	F	T	T

- It is also possible to show this equivalence using the properties of Boolean algebra.
- Explicitly, by definition, $\neg(A \vee \neg B) \Rightarrow \neg B$ is equivalent to $\neg\neg(A \vee \neg B) \vee (\neg B)$, which is in turn equivalent to $A \vee (\neg B \vee (\neg B)) = A \vee \neg B$, and this last expression is equivalent to $B \Rightarrow A$.

(c) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ and $P \Rightarrow (P \Leftrightarrow Q)$.

- We form a truth table to see the statements are not equivalent:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$	$P \Leftrightarrow Q$	$P \Rightarrow (P \Leftrightarrow Q)$
T	T	T	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	F	T
F	F	T	F	T	T	T

- Explicitly, if P is true and Q is false, then the first statement is true while the second is false.

(d) $\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R)$ and True.

- We form a truth table to see they are in fact equivalent.

P	Q	R	$\neg(\neg P \wedge Q)$	$P \vee R$	$Q \wedge \neg R$	$\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R)$
T	T	T	T	T	F	T
T	T	F	T	T	T	T
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	T	F	F	F	T	T
F	F	T	T	T	F	T
F	F	F	T	F	F	T

- Alternatively, using Boolean algebra properties, we have $\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R) = (\neg\neg P \vee \neg Q) \vee (P \vee R) \vee (Q \wedge \neg R) = (P \vee \neg Q) \vee (P \vee R) \vee (Q \wedge \neg R) = (P \vee \neg Q \vee R) \vee (Q \wedge \neg R) = (P \vee \neg Q \vee R \vee Q) \wedge (P \vee \neg Q \vee R \vee \neg R) = \text{True} \vee \text{True} = \text{True}$, as required.

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7. Suppose $P_1, P_2, \ldots, P_n$ are propositions. Observe that there are 2^n rows in a truth table for $P_1, P_2, \ldots, P_n$.

(a) For any single row R in the truth table, describe a Boolean sentence involving $P_1, P_2, \ldots, P_n$ and the connectives $\neg$ and $\wedge$ that is true in row R and false in all other rows.

- Construct the sentence $(\star P_1) \wedge (\star P_2) \wedge \cdots \wedge (\star P_n)$ where $\star$ is nothing if P_i is true in row R and is $\neg$ if P_i is false in row R.
- The sentence is only true when all of the individual components are true, and this occurs precisely when all of the P_i have the truth value given in row R.

(b) For any assignment of true and false values to the 2^n rows in the truth table, show that there exists a Boolean sentence involving $P_1, \ldots, P_n$ and the connectives $\neg$, $\wedge$, and $\vee$ taking the assigned value in each row. [Hint: Join sentences from (a) with $\vee$.]

- By (a) for any row we can construct a sentence that is true in only that row.
- If we list these sentences for all of the rows we seek to have the value True and then connect each of these sentences by $\vee$, then the resulting sentence is true if and only if at least one of the individual sentences is true, which happens if and only if we are in a row that is supposed to have the value True.

(c) Define the connective $\uparrow$ as $P \uparrow Q = \neg(P \wedge Q)$. Show that $P \vee Q$ is logically equivalent to $(P \uparrow P) \uparrow (Q \uparrow Q)$.

- We form a truth table to check the equivalences:

P	Q	$P \uparrow P$	$Q \uparrow Q$	$(P \uparrow P) \uparrow (Q \uparrow Q)$	$P \vee Q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

- Alternatively, by definition we have $P \uparrow P = \neg(P \wedge P) = \neg P$ and in the same way $Q \uparrow Q = \neg Q$, so $(P \uparrow P) \uparrow (Q \uparrow Q) = \neg(\neg P \wedge \neg Q) = (\neg\neg P) \vee (\neg\neg Q) = P \vee Q$.

(d) With $\uparrow$ defined as in part (c), show that $\neg P$ and $P \wedge Q$ can also be expressed as statements involving only the connective $\uparrow$.

- As seen in (c) we have $\neg P = P \uparrow P$, and then $P \wedge Q = \neg(P \uparrow Q) = (P \uparrow Q) \uparrow (P \uparrow Q)$.

(e) Deduce that for any assignment of true and false values to the 2^n rows in the truth table, show that there exists a Boolean sentence involving only $P_1, \ldots, P_n$ and the connective $\uparrow$ taking the assigned value in each row.

- By (b) we can construct such a sentence involving only $P_1, \ldots, P_n$ and the connectives $\neg$, $\wedge$, and $\vee$. By (c) and (d) we can convert any statement involving $\neg$, $\wedge$, or $\vee$ into one only involving $\uparrow$.
- Applying this conversion to the sentence supplied by (b) then yields a Boolean sentence involving $P_1, \ldots, P_n$ and $\uparrow$ taking the assigned values.
