

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Identify whether each of the following sets is finite, countably infinite, or uncountably infinite:
  - (a) The set of real numbers that can be described using at most 60 characters written in English. (For example, 1 can be described as “one” while  $\pi$  can be described as “the ratio of a circle’s circumference to its diameter”.)
  - (b) The set of real numbers that can be described using at most 100,000 characters written in English.
  - (c) The set of real numbers that can be described using a finite number of characters written in English.
  - (d) The set of real numbers that cannot be described by any finite string of characters written in English.

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2. For each set  $G$  with the given operation, identify whether or not  $G$  is a group under the operation:
  - (a) The positive real numbers under addition.
  - (b) The positive real numbers under multiplication.
  - (c) The set of all residue classes  $\{\bar{0}, \bar{1}, \dots, \overline{m-1}\}$  modulo  $m$ , under multiplication.
  - (d) The set of invertible residue classes modulo  $m$ , under multiplication.
  - (e) The set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  under pointwise addition:  $(f + g)(x) = f(x) + g(x)$  for each real  $x$ .
  - (f) The set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  under function composition:  $(f \circ g)(x) = f(g(x))$  for each real  $x$ .
  - (g) The set of bijections  $f : \mathbb{R} \rightarrow \mathbb{R}$  under function composition:  $(f \circ g)(x) = f(g(x))$  for each real  $x$ .

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3. Compute / find each of the following things related to groups:
  - (a) The product  $(sr^2)(r^5)$  in the dihedral group  $D_{2 \cdot 5}$ .
  - (b) The product  $(sr^2)(sr^3)(r^2)(s)$  in the dihedral group  $D_{2 \cdot 6}$ .
  - (c) The cycle decomposition of the permutation  $\sigma \in S_8$  with  $\sigma(1) = 4$ ,  $\sigma(2) = 8$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 3$ ,  $\sigma(5) = 2$ ,  $\sigma(6) = 7$ ,  $\sigma(7) = 6$ , and  $\sigma(8) = 1$ .
  - (d) The cycle decomposition of the permutation  $\sigma \in S_8$  with  $\sigma(n) \equiv n + 2 \pmod{8}$  for all  $n$ .
  - (e) The cycle decomposition of the permutation  $\sigma \in S_8$  with  $\sigma(n) \equiv 3n \pmod{8}$  for all  $n$ .
  - (f) Give an example of an abelian group of order 6.
  - (g) Give an example of a non-abelian group of order 6.
  - (h) Give an example of a non-abelian group of order 24.

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear arguments.

4. Let  $A = (0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ ,  $B = [3, 5] = \{x \in \mathbb{R} : 3 \leq x \leq 5\}$ , and  $C = [1, 6) = \{x \in \mathbb{R} : 1 \leq x < 6\}$ .
    - (a) Show that there exists a bijection between  $A$  and  $B$ .
    - (b) Show that there exists a bijection between  $A$  and  $C$ .
    - (c) Show that there exists a bijection between  $B$  and  $C$ .
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5. The goal of this problem is to give another proof that  $\mathbb{R}$  is uncountable. Let  $f : \mathcal{P}(\mathbb{Z}_+) \rightarrow \mathbb{R}$  be the function defined as follows:  $f(A)$  is the decimal whose  $n$ th decimal place is 1 if  $n \in A$  and is 2 if  $n \notin A$ .
- Find the decimal expansion to 10 digits of the value of  $f$  on each of these sets: (i) the set of even integers, (ii) the set of odd integers, and (iii) the set of prime numbers.
  - Prove that  $f$  is one-to-one. [Hint: Use the fact that  $f(A)$  has a unique decimal expansion for any set  $A$ .]
  - Using the fact that  $\mathcal{P}(\mathbb{Z}_+)$  is uncountable, show that  $\mathbb{R}$  is uncountable.
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6. The goal of this problem is to give yet another proof that  $\mathbb{Q}$  is countable.

- Let  $f : \mathbb{Q} \rightarrow \mathbb{Z}_+$  be the map defined by  $f((-1)^k a/b) = 2^k 3^a 5^b$ , where  $a \geq 0$ ,  $b > 0$ ,  $a/b$  is in lowest terms, and  $k \in \{0, 1\}$ . Show that  $f$  is one-to-one.
  - Show that there exists a bijection between  $\mathbb{Q}$  and  $\mathbb{Z}_+$ . [Hint: Use Cantor-Schröder-Bernstein with  $g : \mathbb{Z}_+ \rightarrow \mathbb{Q}$  defined by  $g(n) = n$ .]
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7. The goal of this problem is to show that there exists a bijection between  $\mathbb{R}$  and  $\mathbb{R} \times \mathbb{R}$ .

- Give an example of a one-to-one function  $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$ .
- Consider the function  $g : [0, 1] \times [0, 1] \rightarrow [0, 1]$  where  $g(0.d_1 d_2 d_3 \dots, 0.e_1 e_2 e_3 \dots) = 0.1 d_1 e_1 1 d_2 e_2 1 d_3 e_3 \dots$ , where we always choose the decimal expansion ending in a string of 9s if there is a choice. Show that  $g$  is one-to-one.
- Deduce that there exists a bijection between  $[0, 1]$  and  $[0, 1] \times [0, 1]$ .
- Show that there exists a bijection between  $[0, 1]$  and  $\mathbb{R}$ . [Hint: Use  $f(x) = x$  and  $g(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$ .]
- Deduce that there exists a bijection between  $\mathbb{R}$  and  $\mathbb{R} \times \mathbb{R}$ . [Hint: Apply (d) to each copy of  $[0, 1]$  in (c).]

**Remark:** Some other results related to these are (i) there exists a *continuous* onto function  $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$  (such functions are called “space-filling curves”), but (ii) there does not exist a continuous bijection  $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$ . Also, (iii) there exists an additive bijection  $f : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  such that  $f(a + b) = f(a) + f(b)$  for all  $a, b \in \mathbb{R}$ .

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8. A real number  $r$  is algebraic if it is a root of a nonzero polynomial with integer coefficients, meaning that  $p(r) = 0$  for some polynomial  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  where  $a_n, a_{n-1}, \dots, a_0$  are integers with  $a_n \neq 0$ . We say  $r$  is transcendental if it is not algebraic. For example,  $3/4$  is algebraic because it is a root of  $p(x) = 3x - 4$ , and  $\sqrt{2}$  is algebraic because it is a root of  $p(x) = x^2 - 2$ .

- Let  $P_n$  be the set of nonzero polynomials of degree at most  $n$  whose coefficients are integers that are at most  $n$  in absolute value. Show that the cardinality of  $P_n$  is finite.
- Let  $S_n$  be the set of roots of the polynomials in the set  $P_n$  defined in part (a). Show that  $S_n$  is finite and that the set  $A$  of algebraic numbers is the union  $\cup_{n \geq 1} S_n$ . [Hint: Use the fact that a nonzero polynomial of degree  $n$  has at most  $n$  different roots.]
- Show that the set of algebraic numbers is countable. Deduce that there are uncountably many transcendental numbers.

**Remark:** The result of (c) is what is termed “non-constructive”: it proves that transcendental numbers exist without actually giving any examples. In fact, it is usually quite difficult to prove that any specific real number  $r$  actually is transcendental.

**Remark:** The ideas from this problem can be adapted to show that the cardinality of the “computable numbers” (real numbers that can be computed to arbitrarily good accuracy by a finite, terminating algorithm) is countable, and thus that there are uncountably many uncomputable numbers.

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