E. Dummit's Math 1465 $\sim$ Intensive Mathematical Reasoning, Fall 2024 $\sim$ Homework 1, due Tue Sep 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Match the erroneous proofs (a)-(e) with the reasons (1)-(5) they are not valid proofs of the claims.

   Proofs:

   (a) Proposition: The integer 2 is odd.
   Proof: It is a fact about odd integers that any odd integer plus any odd integer always gives an even integer. Because $2 + 2 = 4$, and 4 is an even integer, this means 2 must be odd.

   (b) Proposition: For any proposition P, $\neg(\neg(\neg P)))$ is always false.
   Proof: If P is true, then $\neg(\neg(\neg P)))$ is false, and if P is false then $\neg(\neg(\neg P)))$ is also false.

   (c) Proposition: If x is an integer and $3x - 2 = 7$, then $x = 3$.
   Proof: Suppose $x = 3$. Then $3x - 2 = 3(3) - 2 = 7$. Therefore, if $3x - 2 = 7$, then $x = 3$.

   (d) Proposition: Every odd number greater than 1, except for 9, is prime.
   Proof: Clearly, 3 is prime, 5 is prime, 7 is prime, 9 is not prime, 11 is prime, and 13 is prime. Since we have excluded 9, all odd numbers greater than 1 are prime.

   (e) Proposition: For an integer m, m is even if and only if m^2 is even.
   Proof: Suppose m is even. Then $m = 2k$ for some integer k, meaning that $m^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$ and thus m^2 is even. Therefore, m is even if and only if m^2 is even.

   Reasons:

   (1) The proof actually shows the converse of the required statement.
   (2) The proof is a list of examples only showing the result in some cases, not all cases.
   (3) The proof only shows one direction of a biconditional.
   (4) The proof is a list of examples where not all the examples are correct.
   (5) The proof references a true statement but this statement does not imply the conclusion.

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2. Find a counterexample to each of the following statements (make sure to explain briefly why your example is a counterexample):

   (a) If n is any integer, then n^2 is greater than n.
   (b) If a and b are positive integers, then $ab + a + b \neq 7$.
   (c) If P and Q are any propositions, then $(\neg P \wedge \neg Q) \vee (P \wedge Q)$ is always true.
   (d) If P and Q are any propositions, then $\neg P \wedge Q$ is logically equivalent to $\neg(P \vee Q)$.
   (e) There is no integer n such that both n and $3n + 1$ are prime numbers.
   (f) If p is a prime number, then $2^p - 1$ is also a prime number.

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3. Write explicitly the converse, inverse, and contrapositive of the following conditional statements:

   (a) If you do not study for your exams, then you will get bad grades.
   (b) If n is an odd integer greater than 7, then n is the sum of three odd primes.
   (c) If you want to bake a cake, then you must have eggs and flour.

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4. Answer the following questions based on the "Tips on Problem Solving" course page: https://dummit.cos.northeastern.edu/teaching_fa24_1465/1465_problemsolving.html

   (a) What are five different general tips on problem solving presented at the link?
   (b) In Math 1465, what can you do if you are having trouble getting started on a problem?
   (c) Can you solve a problem that asks you to prove something by listing examples and explaining them?
   (d) How can you establish that two statements are equivalent (A if and only if B)?
   (e) How can you show that two numerical expressions are equal?
   (f) How can you show that an object is unique?
   (g) In Math 1465, how can you tell whether your proof has any gaps, or needs more detail?
   (h) Should your proofs be written in full sentences? Easy to read? Clear and concise?

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Suppose that P, Q, and R are any propositions.

   (a) Prove that if P and $P \Rightarrow Q$ are both true, then Q is also true. [Hint: Use a truth table to identify all cases in which both P and $P \Rightarrow Q$ are true.]
   (b) Suppose that the statements "If it is raining, then it is cloudy" and "It is raining" are both true. Is the statement "It is cloudy" necessarily true? Explain. [Hint: Use (a).]
   (c) Prove that if $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is also true.
   (d) Suppose that the statements "If it is raining, then it is cloudy" and "If it is cloudy, then people want to stay home" are both true. Is the statement "If it is raining, then people want to stay home" necessarily true? Explain. [Hint: Use (c).]

   Remarks: The results of (b) and (d) here are forms of reasoning we often apply in deductive proofs (formally, (b) is known as *modus ponens* and (d) is known as *transitivity*). The point of this problem is to verify that these forms of deduction are in fact logically correct.

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6. Using a truth table or otherwise, determine whether each of following pairs of statements are equivalent. For those that are false, give an explicit counterexample (i.e., truth values for the propositions showing the statements are different):

   (a) $A \wedge (A \vee B)$ and A.
   (b) $\neg(A \vee \neg B) \Rightarrow \neg B$ and $B \Rightarrow A$.
   (c) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ and $P \Rightarrow (P \Leftrightarrow Q)$.
   (d) $\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R)$ and True.

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7. Suppose $P_1, P_2, \ldots, P_n$ are propositions. Observe that there are 2^n rows in a truth table for $P_1, P_2, \ldots, P_n$.

   (a) For any single row R in the truth table, describe a Boolean sentence involving $P_1, P_2, \ldots, P_n$ and the connectives $\neg$ and $\wedge$ that is true in row R and false in all other rows.
   (b) For any assignment of true and false values to the 2^n rows in the truth table, show that there exists a Boolean sentence involving $P_1, \ldots, P_n$ and the connectives $\neg$, $\wedge$, and $\vee$ taking the assigned value in each row. [Hint: Join sentences from (a) with $\vee$.]
   (c) Define the connective $\uparrow$ as $P \uparrow Q = \neg(P \wedge Q)$. Show that $P \vee Q$ is logically equivalent to $(P \uparrow P) \uparrow (Q \uparrow Q)$.
   (d) With $\uparrow$ defined as in part (c), show that $\neg P$ and $P \wedge Q$ can also be expressed as statements involving only the connective $\uparrow$.
   (e) Deduce that for any assignment of true and false values to the 2^n rows in the truth table, show that there exists a Boolean sentence involving only $P_1, \ldots, P_n$ and the connective $\uparrow$ taking the assigned value in each row.
