1. Calculate/determine the following things:

   (a) With universe $\mathbb{Z}_+$, let $E(n)$ = "n is even" and $S(n)$ = "n is a perfect square greater than 1". Give a counterexample to the statement $\forall n\ [E(n) \wedge (n > 2)] \Rightarrow [\exists m\, S(m) \wedge m|n]$.

   (b) Give a counterexample to this statement: If $n \neq 3$ then $n^2 \neq 9$.

   (c) Negate this statement: $\forall x \forall y \exists z,\ x + y + z > 5$.

   (d) Negate this statement: $\forall x \in A\, \forall y \in B,\ x \cdot y \in A \cap B$.

   (e) Negate this statement: for any $x \in \mathbb{R}$ there exists an $n \in \mathbb{Z}$ such that $x < n$.

   (f) If $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 4, 8\}$, find $A \times (A \cap B)$.

   (g) Find subsets A, B of a universal set $\{1, 2, 3, 4\}$ giving a counterexample to the statement $(A \cap B)^c \cup B \subseteq (A^c \cup B)^c$.

   (h) With universe $\mathbb{R}$, find truth values of (i) $\forall x \forall y\ y \neq x$, (ii) $\forall x \exists y\ y \neq x$, (iii) $\exists x \forall y\ y \neq x$, (iv) $\exists x \exists y\ y \neq x$.

   (i) With universe $\mathbb{R}$, find truth values of (i) $\forall x \forall y\ y^2 \geq x$, (ii) $\forall x \exists y\ y^2 \geq x$, (iii) $\exists x \forall y\ y^2 \geq x$, (iv) $\exists x \exists y\ y^2 \geq x$.

   (j) Give a counterexample to this statement: For integers a, b, and c, if $a|b$ and $a|c$, then $b|c$.

   (k) Find the gcd and lcm of 256, 520 and also of 921, 177.

   (l) Find the gcd and lcm of $2^3 3^2 5^4 7$ and $2^4 3^3 5^4 11$.

   (m) Give a counterexample to this statement: If p and q are prime, then $p + q$ is never prime.

   (n) Give a counterexample to this statement: If $n > 1$ is an integer, then $\sqrt{n}$ is always irrational.

   (o) Negate this statement: there exist positive integers a and b with $2 = (a/b)^3$.

   (p) Find the values of $\overline{4} + \overline{8}$, $\overline{4} - \overline{8}$, $\overline{4} \cdot \overline{8}$, $\overline{4}^2$, and $\overline{4}^{-1}$ in $\mathbb{Z}/9\mathbb{Z}$.

   (q) Identify which of $\overline{10}$, $\overline{11}$, $\overline{12}$ mod 25 have multiplicative inverses and find the inverses of those that do.

   (r) Identify which of $\overline{30}$, $\overline{31}$, $\overline{32}$ mod 42 have multiplicative inverses and find the inverses of those that do.

   (s) List the ordered pairs in the equivalence relation on $\{1, 2, 3, 4, 5, 6\}$ with corresponding partition $\{1, 2, 4\}, \{3, 5\}, \{6\}$.

   (t) Find a formula for the inverse of the function $f : \mathbb{Q}\backslash\{\frac{7}{2}\} \to \mathbb{Q}$ where $f(x) = \frac{6x+5}{2x-7}$.

   (u) Define $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ via $f = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Find $\text{im}(f)$. Is f one-to-one? Onto?

   (v) Define $g : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ via $g(\overline{n}) = \overline{2n + 3}$. Find $\text{im}(g)$. Is g one-to-one? Onto?

   (w) Define $h : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ via $h(\overline{n}) = \overline{5n - 3}$. Find $\text{im}(h)$. Is h one-to-one? Onto?

   (x) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 2x$. Is f one-to-one? Onto? What is f^{-1}?

   (y) Define $g : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = 2x$. Is f one-to-one? Onto? What is g^{-1}?

   (z) Find functions $f, g : \mathbb{Z} \to \mathbb{Z}$ with f one-to-one but not onto and g onto but not one-to-one.

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2. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) irreflexive, (vi) an equivalence relation, (vii) a partial ordering, and (viii) a total ordering.

   (a) $R = \{(1, 1), (2, 1), (2, 2)\}$ on the set $\{1, 2\}$.

   (b) $R = \{(1, 2), (2, 1)\}$ on the set $\{1, 2\}$.

   (c) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ on the set $\{1, 2, 3, 4\}$.

   (d) The divisibility relation on the set $\{2, -3, 4, -5, 6\}$.

   (e) The divisibility relation on the set $\{2, -4, -12, 36\}$.

   (f) The relation R on $\mathbb{Z}$ with $a\ R\ b$ precisely when $|a| \equiv |b|$ modulo 5.

   (g) The relation R on $\mathbb{R}$ with $a\ R\ b$ precisely when $ab > 0$.

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3. Calculate/determine the following things:

(a) Which of the sets $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q} \times \mathbb{R}$, $\mathbb{Q} \cap \mathbb{R}$, $\mathbb{Q} \cup \mathbb{R}$, and $\mathbb{R} \backslash \mathbb{Q}$ are countable.

(b) Which of the power sets $\mathcal{P}(\emptyset)$, $\mathcal{P}(\{1, 2, 3, 4, 5, \dots, 10000\})$, $\mathcal{P}(\mathbb{Z})$, $\mathcal{P}(\mathbb{Q})$, $\mathcal{P}(\mathbb{R})$, and $\mathcal{P}(\mathcal{P}(\mathbb{Q}))$ are countable.

(c) Whether the set $(\mathbb{Z}, -)$ of integers under subtraction is a group.

(d) Whether the set $(2\mathbb{Z}, +)$ of even integers under addition is a group.

(e) The inverse of $\overline{30}$ in the additive group $\mathbb{Z}/59\mathbb{Z}$ of integers modulo 59.

(f) The inverse of $\overline{30}$ in the multiplicative group $(\mathbb{Z}/59\mathbb{Z})^{\times}$ of nonzero integers modulo 59.

(g) The products $(sr)(sr^2)$ and $s^2r^3s^4r^5$ in the dihedral group $D_{2\cdot 10}$ of order 20.

(h) The inverses of r^3 and sr^2 in the dihedral group $D_{2\cdot 12}$ of order 24.

(i) The cycle decomposition of the permutation $\sigma \in S_8$ with $\sigma(n) = 9 - n$ for each $1 \leq n \leq 8$.

(j) The cycle decomposition of the permutation $\sigma \in S_7$ with $\sigma(n) \equiv 3n \pmod 7$ for each $1 \leq n \leq 7$.

(k) The cycle decomposition of the product $(3\,1\,4)(1\,5)$ in S_6.

(l) The cycle decomposition of the product $(2\,7\,1\,8) \cdot (2\,8) \cdot (1\,8) \cdot (2\,8)$ in S_8.

(m) The cycle decomposition of the inverse of $(1\,4\,2\,8\,5)(6\,7)$ in S_8.

(n) An example of an abelian group of order 8.

(o) An example of a non-abelian group of order 20.

(p) A example of a countably infinite abelian group.

(q) An example of a countably infinite non-abelian group.

(r) An example of an uncountably infinite abelian group.

(s) An example of an uncountably infinite non-abelian group.

(t) The orders of the elements s, r, and r^2 in the dihedral group $D_{2\cdot 10}$.

(u) The orders of the elements $(1\,2\,3)$, $(4\,5)$, and $(1\,2\,3)(4\,5)$ in S_5.

(v) The orders of the elements $(2\,4\,5)$, $(1\,5)(2\,3)$, and $(2\,4\,5) \cdot (1\,5)(2\,3)$ in S_5.

(w) The possible orders of a subgroup of a group of order 20.

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4. Prove the following:

(a) Let P, Q, and R be propositions. Show that $P \wedge \neg[Q \vee (R \Rightarrow P)]$ is always false.

(b) Let P, Q, and R be propositions. Show that $(P \Rightarrow Q) \Leftrightarrow R$ and $P \Rightarrow (Q \Leftrightarrow R)$ are not logically equivalent.

(c) Let P, Q, and R be propositions. Show that $\neg[Q \wedge \neg(P \wedge Q)] \wedge \neg P$ and $\neg Q \wedge \neg P$ are logically equivalent.

(d) Let A, B, and C be sets. Prove that $(A \backslash B) \cup (B \backslash C) \subseteq (A \cup B) \backslash (B \cap C)$.

(e) Let A and B be sets. Prove that $A \backslash B = \emptyset$ if and only if $A \subseteq B$.

(f) For any sets A, B, C, prove $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

(g) For any sets A, B inside a universal set U, prove $A \cup B^c = U$ if and only if $A^c \cap B = \emptyset$.

(h) For any sets A, B, C, prove $A \subseteq B \cup C$ if and only if $A \backslash B \subseteq C$.

(i) If F_n is the nth Fibonacci number, show $F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$ for every positive integer n.

(j) Suppose $c_1 = c_2 = 2$, and for all $n \geq 3$, $c_n = c_{n-1}c_{n-2}$. Prove that $c_n = 2^{F_n}$ for every positive integer n, where F_n is the nth Fibonacci number.

(k) Suppose $a_1 = 1$ and $a_n = 3a_{n-1} + 4$ for all $n \geq 2$. Prove that $a_n = 3^n - 2$ for every positive integer n.

(l) Suppose $b_1 = 3$ and $b_{n+1} = 2b_n - n + 1$ for all $n \geq 2$. Prove that $b_n = 2^n + n$ for every positive integer n.

(m) A sequence is defined by the recurrence relation $c_n = 4c_{n-1} - 4c_{n-2}$ for $n \geq 2$, where $c_0 = 6$ and $c_1 = 8$. Prove that $c_n = (6 - 2n)2^n$ for all integers $n \geq 0$.

(n) Suppose $d_1 = 2$, $d_2 = 4$, and for all $n \geq 3$, $d_n = d_{n-1} + 2d_{n-2}$. Prove that $d_n = 2^n$ for every positive integer n.

(o) Show that $25^n + 7$ is a multiple of 8 for every positive integer n.

(p) Prove that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for every positive integer n.

(q) Prove that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$ for every positive integer n.

(r) Prove that the sum of any six consecutive integers is congruent to 3 modulo 6.

(s) Prove that $7^n + 5$ is divisible by 6 for all positive integers n.

(t) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Show that $a(b + c) \equiv b(a + d) \pmod{n}$.

(u) Suppose n is an integer. Prove that $2|n$ and $3|n$ if and only if $6|n$.

(v) If $A = \{4a + 6b : a, b \in \mathbb{Z}\}$ and $B = \{2c : c \in \mathbb{Z}\}$, prove that $A = B$.

(w) If p is a prime, prove that $\gcd(n, n + p) > 1$ if and only if $p|n$.

(x) If $C = \{6c : c \in \mathbb{Z}\}$ and $D = \{10a + 14b : a, b \in \mathbb{Z}\}$, prove that $C \subseteq D$.

(y) Prove that the product of two consecutive even integers is always 1 less than a perfect square.

(z) If n is any positive integer, prove that $n - 1$ is invertible modulo n and its multiplicative inverse is itself.

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5. Prove the following:

(a) If $R, S : A \to B$ are relations, prove that $R^{-1} \cap S^{-1} = (R \cap S)^{-1}$.

(b) Show $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| = |y|\}$ is an equivalence relation on $\mathbb{Z}$ and list the equivalence classes of 0, 2, -2, 4.

(c) A relation R on integers is defined via $x\, R\, y$ when $5|(6x - y)$. Prove that R is an equivalence relation and describe the equivalence classes of R.

(d) Show that the only equivalence relation R on A that is a function from A to A is the identity relation.

(e) Suppose R is a reflexive and transitive relation on a set A. Show that $S = R \cap R^{-1}$ is an equivalence relation on A.

(f) Suppose that $f : A \to A$ is a function. Show that $f(f(a)) = a$ for all $a \in A$ if and only if $f^{-1} : A \to A$ is a function and $f^{-1}(a) = f(a)$ for all $a \in A$.

(g) Suppose $f : B \to C$ is one-to-one. If $g, h : A \to B$ have $f \circ g = f \circ h$, show that $g = h$.

(h) Let $f : A \to B$ be a function. For a subset D of B, recall that $f^{-1}(D) = \{a \in A : f(a) \in D\}$. For any subset C of A, show that $C \subseteq f^{-1}(f(C))$.

(i) If $f : A \to B$ is one-to-one and C is a subset of A, prove that $C = f^{-1}(f(C))$.

(j) If $f : A \to B$ is a function and D is any subset of B, show that $f(f^{-1}(D)) \subseteq D$.

(k) If $f : A \to B$ is onto and D is any subset of B, prove that $f(f^{-1}(D)) = D$.

(l) If $f : A \to B$ is one-to-one and C_1, C_2 are subsets of A, prove that $f(C_1) \cap f(C_2) \subseteq f(C_1 \cap C_2)$.

(m) Suppose $f : A \to B$ is a bijection. Show that $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ given by $\tilde{f}(S) = \{f(s) : s \in S\}$ is also a bijection, where $\mathcal{P}(S)$ denotes the power set of S (the set of subsets of S).

(n) Let S be the set of equivalence classes of an equivalence relation R on A and define the function $f : A \to S$ via $f(a) = [a]$. Show that f is one-to-one if and only if R is the identity relation.

(o) Prove that if A is countable and B is uncountable, then the set difference $B \backslash A$ is uncountable.

(p) Prove that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$, the set of rational numbers strictly between 0 and 1.

(q) Prove that the set $\mathbb{Q} \times \mathbb{Z}$ is countable and that the set $\mathbb{R} \times \mathbb{Z}$ is uncountable.

(r) Prove that the set of all finite subsets of $\mathbb{Z}$ is countable.

(s) Prove that there exists a bijection between the half-closed interval $[1, 7) = \{x \in \mathbb{R} : 1 \leq x < 7\}$ and the open interval $(2, 9) = \{x \in \mathbb{R} : 2 < x < 9\}$.

(t) Prove that there exists a bijection between $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ and $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

(u) Suppose g and h are elements in a group such that $gh = hg$. Prove that $gh^n = h^n g$ for all positive integers n.

(v) Suppose g and h are elements of a group such that $g^{-1}h^{-1} = h^{-1}g^{-1}$. Prove that $gh = hg$.

(w) Suppose g is an element of a group G and that g has order n. Show that $g^{-1} = g^{n-1}$.

(x) Let $f : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$ be a bijection. Show that there exists some positive integer A such that $f^A(i) = i$ for each $i \in \{1, 2, 3, \ldots, n\}$, where f^A denotes f composed with itself A times.

(y) Suppose G is a group and H is a subgroup, and define the relation R on G by saying $g_1\, R\, g_2$ whenever there exists $h \in H$ such that $g_1 = hg_2$. Prove that R is an equivalence relation.

(z) Suppose G is an abelian group. Show that the subset $S = \{a \in G : a^2 = e\}$ is a subgroup of G.
