Lecture: Monday-Thursday 6:00pm–7:40pm, Hayden Hall 321.

Instructor: Evan Dummit, Lake Hall 571, edummit@northeastern.edu.

Office Hours: Wednesday 3:00pm-4:00pm, or by appointment, 571 Lake Hall.

Course Webpage: https://web.northeastern.edu/dummit/teaching_fa23_7359.html.

Course Textbooks: The instructor will write course notes in lieu of an official textbook. Silverman's "Arithmetic of Elliptic Curves" and Diamond/Shurman's "A First Course in Modular Forms" are good references for the material. Serre's "A Course in Arithmetic" and Koblitz's "Introduction to Elliptic Curves and Modular Forms" are also good references for some of the material, although they present it differently.

Expected Background: There are no formal prerequisites, but students should have comfort with algebra at the beginning graduate level (Math 5111 or 5112) and complex analysis at the undergraduate level (Math 4555). I will freely refer to some results from commutative algebra, algebraic geometry, and complex analysis, but the goal is to make the course as self-contained as possible.

Course Overview: This course is a survey of elliptic curves and modular forms, which constitute two major areas of modern research in number theory. Elliptic curves are algebraic curves of genus 1 that arise in a wide variety of contexts in mathematics and their study involves techniques from nearly every discipline: algebra, analysis, geometry, topology, and (of course) number theory. Modular forms are analytic functions on the complex upper half-plane satisfying a certain functional equation, and although they are intrinsically analytic objects, they turn out to have surprisingly deep connections to the (seemingly far more) algebraically-flavored elliptic curves. The connection between elliptic curves and modular forms, which falls under the broader heading of "modularity", is central to Wiles's proof of Fermat's Last Theorem, and one of the end goals of the course is to elucidate some of the major ideas of this connection, along with highlighting other modern developments in number theory related to elliptic curves and modular forms.

Course Topics: We begin with an overview of the addition law on elliptic curves, and then reformulate these ideas in the language of divisors using the Riemann-Roch theorem. We then study general structural properties of elliptic curves over arbitrary fields: isogenies, differentials, the Tate module, the Weil pairing, endomorphisms, and automorphisms. Next, we study elliptic curves over two more specialized types of fields: over finite fields $\mathbb{F}_q$, which involves more number-theoretic techniques, and over the complex numbers $\mathbb{C}$, which involves complex-analytic and geometric techniques. We then review some of the classical theory of modular forms with a primary aim of discussing modularity of elliptic curves, focusing in particular on the relationship between the L-function of an elliptic curve and modular forms, which will tie together essentially all of the material from the course. Time permitting, we will discuss some of the modern developments in the study of elliptic curves and modular forms (e.g., Wiles's proof of Fermat's Last Theorem, the Birch and Swinnerton-Dyer conjecture and work of Gross/Zagier and Kolyvagin, Bhargava's work on average n-Selmer ranks, or other topics of student or instructor interest).

Grades: Grades will be based on attendance and participation (50%), and on occasional homework assignments (50%), which will be collaborative and may involve roundtable discussions of the solutions. Depending on student and instructor interest, there may also be student presentations on course-related topics during the semester.

Course Schedule: The course is tentatively organized as follows:

Weeks 1-5: General Properties of Elliptic Curves: cubic curves and elliptic curves, Weierstrass equations and the group law, divisors and Riemann-Roch, genus-1 curves, isogenies, differentials, the Weil pairing, endomorphisms and automorphisms.

Weeks 6-7: Elliptic Curves over $\mathbb{F}_q$: elliptic curves over finite fields, the Hasse bound, the Weil conjectures, quaternion algebras and endomorphisms over $\mathbb{F}_q$, Schoof's algorithm.

Weeks 8-10: Elliptic Curves over $\mathbb{C}$: elliptic curves as Riemann surfaces, complex lattices and complex tori, elliptic functions, the Weierstrass $\wp$-function, uniformization, elliptic curves with complex multiplication.

Weeks 11-14: Modular Forms and L-functions: modular functions and modular forms, congruence subgroups, Hecke operators, L-functions of elliptic curves, modularity, Fermat's last theorem, modern developments.

Attendance Policy: It is expected that you will attend every class (see above under "Grades"), but I am willing to work with you if you will have to miss some of the lectures. Since this is a graduate-level topics course, it is very easy to get lost even if you only miss one day.

If you will be absent from a class activity due to a religious observance or practice, or for participation in a university-sanctioned event (e.g., university athletics), it is your responsibility to inform the instructor during the first week of class and provide appropriate documentation if required.

Statement on Academic Integrity: A commitment to the principles of academic integrity is essential to the mission of Northeastern University. Academic dishonesty violates the most fundamental values of an intellectual community and undermines the achievements of the entire University. Violations of academic integrity include (but are not limited to) cheating on assignments or exams, fabrication or misrepresentation of data or other work, plagiarism, unauthorized collaboration, and facilitation of others' dishonesty. Possible sanctions include (but are not limited to) warnings, grade penalties, course failure, suspension, and expulsion.

Statement on Accommodations: Any student with a disability is encouraged to meet with or otherwise contact the instructor during the first week of classes to discuss accommodations. The student must bring a current Memorandum of Accommodations from the Office of Student Disability Services.

Statement on Classroom Behavior: Disruptive classroom behavior will not be tolerated. In general, any behavior that impedes the ability of your fellow students to learn will be viewed as disruptive.

Statement on Inclusivity: Faculty are encouraged to address students by their preferred name and gender pronoun. If you would like to be addressed using a specific name or pronoun, please let your instructor know.

Statement on Evaluations: Students are requested to complete the TRACE evaluations at the end of the course.

Miscellaneous Disclaimer: The instructor reserves the right to change course policies, including the evaluation scheme of the course. Notice will be given in the event of any substantial changes.