Math 7359 (Elliptic Curves and Modular Forms)

Lecture #20 of 24 $\sim$ November 20, 2023

Elliptic Functions and the Weierstrass $\wp$-Function

- Elliptic Functions
- The Weierstrass $\wp$-Function
- Elliptic Curves and Elliptic Functions

Elliptic Functions, I

Definition

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. An elliptic function relative to Λ is a meromorphic function on $\mathbb{C}$ that satisfies $f(z+\omega) = f(z)$ for all $\omega \in \Lambda$ and $z \in \mathbb{C}$.

The set of all elliptic functions relative to Λ is denoted $\mathbb{C}(\Lambda)$.

When the lattice Λ is clear from context, or not relevant, we will simply say "elliptic function" without explicitly saying "relative to Λ".

- Elliptic functions are also commonly called doubly-periodic functions since the general condition above is equivalent to $f(z+\omega_1) = f(z+\omega_2) = f(z)$: in other words, saying that f has two different periods ω_1 and ω_2.

Elliptic Functions, II

Obviously, constant functions are elliptic functions. Traditionally at this point I would now give some examples. But...

- Keep in mind the general principle that elliptic functions will correspond to rational functions on the associated elliptic curve E.
- So we should expect it to be somewhat challenging to construct elliptic functions, since most functions on E will not be rational.
- We will therefore study general properties of elliptic functions first, and then use the results to give constructions of elliptic functions.

Elliptic Functions, III

As with any meromorphic function, we have notions of the order of vanishing, zeroes, poles, and residues of an elliptic function.

- Explicitly, if f is a nonzero elliptic function on $\mathbb{C}$, then for any $z_0 \in \mathbb{C}$ we have a local Laurent expansion $f(z) = \sum_{n=k}^{\infty} a_n(z - z_0)^n$ at z_0, where we assume the leading coefficient $a_k \neq 0$.
- Note that when $k \geq 0$ this is a familiar power series, while when $k < 0$ this is a Laurent series.
- For this expansion, the order of vanishing of f at z_0, denoted $\mathrm{ord}_{z_0}(f)$, is the value k.
- We say that f has a pole of order $|k|$ at z_0 when $k < 0$ and a zero of order k at z_0 when $k > 0$.
- The residue of f at z_0, denoted $\mathrm{res}_{z_0}(f)$, is the coefficient a_{-1}. Note that the residue can be nonzero only when f has a pole at z_0.

Elliptic Functions, IV

Let us now collect some basic facts about elliptic functions:

Proposition (Properties of Elliptic Functions)

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$, let $\mathbb{C}(\Lambda)$ denote the field of elliptic functions with respect to Λ, and let D be a fundamental region for $\mathbb{C}/\Lambda$ (e.g., the parallelogram with vertices 0, ω_1, ω_2, $\omega_1 + \omega_2$ or some $\mathbb{C}$-translate of it). Then the following hold:

1. *A nonzero elliptic function $f \in \mathbb{C}(\Lambda)$ has finitely many zeroes and poles inside of D.*
2. *An elliptic function with no zeroes, or no poles, is constant.*
3. *For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f) = 0$.*
4. *For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) = 0$.*
5. *For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w \in \Lambda$.*
6. *An elliptic function with at most one pole, with pole order at most 1 there, is constant.*

Elliptic Functions, V

Note to self: get up and write the results on the board.

Elliptic Functions, VI

1. A nonzero elliptic function $f \in \mathbb{C}(\Lambda)$ has finitely many zeroes and poles inside of D.

Proof:

Elliptic Functions, VI

1. A nonzero elliptic function $f \in \mathbb{C}(\Lambda)$ has finitely many zeroes and poles inside of D.

Proof:

- Since the fundamental parallelogram D is compact, if f had infinitely many poles they would have an accumulation point, but poles of a meromorphic function are discrete. Hence f has only finitely many poles.
- Applying the argument to $1/f$ shows that f also has finitely many zeroes, so f has finitely many zeroes and poles.

This is the analogue of the statement that a nonzero rational function in $k(C)$ has only finitely many zeroes and poles.

Elliptic Functions, VII

2. An elliptic function with no zeroes, or no poles, is constant.

Proof:

Elliptic Functions, VII

2. An elliptic function with no zeroes, or no poles, is constant.

Proof:

- If f has no poles then f is holomorphic on all of $\mathbb{C}$ (i.e., f is an entire function).
- Since $\mathbb{C}/\Lambda$ is compact and f is continuous, f is bounded on D, hence on all of $\mathbb{C}$ because f is doubly periodic. But then f is an entire function that is bounded, so by Liouville's theorem, f is constant.
- If f has no zeroes, then applying the same argument to $1/f$ shows that $1/f$ hence f is constant.

This is the analogue of the statement that a rational function in $k(C)$ with no zeroes or no poles is constant.

Elliptic Functions, VIII

3. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f) = 0$.

Proof:

Elliptic Functions, VIII

3. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f) = 0$.

Proof:

- Choose any fundamental region D whose boundary contains no zeroes or poles of f: this is possible since there are only finitely many zeroes and poles by (1), but there are uncountably many inequivalent translations to select for D.
- Consider the integral $\int_{\partial D} f(z)\,dz$: since f takes the same values on parallel edges of ∂D, the contributions to the integral on opposite sides cancel since they have opposite orientations, so the integral is zero.
- Then Cauchy's residue theorem immediately yields $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f) = \dfrac{1}{2\pi i} \int_{\partial D} f(z)\,dz = 0$.

Elliptic Functions, IX

4. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) = 0$.

Proof:

Elliptic Functions, IX

4. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) = 0$.

Proof:

- As in (3), choose any fundamental region D whose boundary contains no zeroes or poles of f.
- Since f is elliptic so is its derivative f' hence so is f'/f.
- If the Laurent series for f at w is $a_k(z-w)^k + \cdots$, then the Laurent series for f' is $ka_k(z-w)^{k-1} + \cdots$ and so the Laurent series for the ratio f'/f is $k(z-w)^{-1} + \cdots$, and so $\mathrm{res}_w(f'/f) = k = \mathrm{ord}_w(f)$.
- Cauchy's residue theorem yields $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) = \sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f'/f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)}\, dz = 0$ since the integral is zero as in (3).

This is the analogue of $\deg(\mathrm{div}\, f) = 0$.

Elliptic Functions, X

5. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w \in \Lambda$.

Discussion:

Elliptic Functions, X

5. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w \in \Lambda$.

Discussion:

- Note that choosing a different fundamental region D will potentially shift points w in the sum by an element of Λ, so unlike the sums in (3) and (4) which are independent of the choice of D, this sum is only well-defined modulo Λ.
- This is the analogue of the statement that a divisor on E is principal iff the underlying sum of points resolves to O.

Proof:

- As in (4), we choose a fundamental region D whose boundary contains no zeroes or poles of f: say with vertices a, $a + \omega_1$, $a + \omega_1 + \omega_2$, $a + \omega_2$ in counterclockwise order.
- By Cauchy's residue theorem we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w = \sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(zf'/f) = \dfrac{1}{2\pi i} \int_{\partial D} z \dfrac{f'(z)}{f(z)}\, dz$.

Elliptic Functions, XI

5. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w \in \Lambda$.

Proof (continued):

- Decomposing the integral into components along the four sides of D, and then applying ellipticity of f'/f yields

$$\begin{aligned}
\int_{\partial D} z \frac{f'(z)}{f(z)}\, dz &= \left[\int_a^{a+\omega_1} + \int_{a+\omega_1}^{a+\omega_1+\omega_2} + \int_{a+\omega_1+\omega_2}^{a+\omega_2} + \int_{a+\omega_2}^{a}\right] z \frac{f'(z)}{f(z)}\, dz \\
&= \int_a^{a+\omega_1} z \frac{f'(z)}{f(z)}\, dz + \int_a^{a+\omega_2} (z+\omega_1) \frac{f'(z)}{f(z)}\, dz \\
&\qquad - \int_a^{a+\omega_1} (z+\omega_2) \frac{f'(z)}{f(z)}\, dz - \int_a^{a+\omega_2} z \frac{f'(z)}{f(z)}\, dz \\
&= -\omega_2 \int_a^{a+\omega_1} \frac{f'(z)}{f(z)}\, dz + \omega_1 \int_a^{a+\omega_2} \frac{f'(z)}{f(z)}\, dz
\end{aligned}$$

Elliptic Functions, XI

5. For any $f \in \mathbb{C}(\Lambda)$, we have $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w \in \Lambda$.

Proof (finished):

- So $\int_{\partial D} z \dfrac{f'(z)}{f(z)}\, dz = -\omega_2 \int_a^{a+\omega_1} \dfrac{f'(z)}{f(z)}\, dz + \omega_1 \int_a^{a+\omega_2} \dfrac{f'(z)}{f(z)}\, dz$
- But now since f'/f is elliptic, we have $(f'/f)(a) = (f'/f)(a+\omega_1)$, so $\int_a^{a+\omega_j} \dfrac{f'(z)}{f(z)}\, dz$ equals $2\pi i$ times the winding number $W_{\gamma_j}(0)$ around 0 of the curve $\gamma_j : [0,1] \to \mathbb{C}$ with $\gamma(t) = f(a + t\omega_j)$.
- Hence we obtain $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) w = \dfrac{1}{2\pi i} \int_{\partial D} z \dfrac{f'(z)}{f(z)}\, dz = -\omega_2 W_{\gamma_1}(0) + \omega_1 W_{\gamma_2}(0)$, which is an element of Λ because the winding numbers are both integers.

Elliptic Functions, XI

6. An elliptic function with at most one pole, with pole order at most 1 there, is constant.

Proof:

- Suppose f were elliptic and had a single simple pole.
- Then by (3), since the sum of the residues of f is 0, the residue at that pole would be zero, but then f would be holomorphic hence constant by (2).

Constructing Elliptic Functions, I

So far we have established some properties of elliptic functions without actually describing any such functions aside from constants. Let us use these properties to (try to) give a construction of an elliptic function.

- From (2) we know that any nonconstant elliptic function must have at least one pole, and from (6) we see that the total pole order must be at least 2.
- Taking motivation from the x-coordinate function on an elliptic curve (which has one pole, of order 2, at ∞), let us try to construct an elliptic function $f(z)$ with a double pole.
- By translation we may place this pole anywhere, so let us put it at 0.

Constructing Elliptic Functions, II

We have a double pole at 0.

- Then the Laurent expansion of $f(z)$ at $z = 0$ is $c_{-2}z^{-2} + O(z^{-1})$ for some $c \neq 0$, and so by rescaling we may assume $c_{-2} = 1$.
- Now, by (3), since f has only one pole (up to periodicity), the residue at that pole must be zero, so the z^{-1} coefficient in the Laurent expansion at $z = 0$ must be zero.
- So in fact, the Laurent expansion for $f(z)$ is of the form $f(z) = z^{-2} + c_0 + c_1 z + c_2 z^2 + \cdots$ for some power series $c_0 + c_1 z + c_2 z^2 + \cdots$ that is necessarily holomorphic in a neighborhood of 0.
- In other words, $f(z) - z^{-2}$ is holomorphic near 0.

Constructing Elliptic Functions, III

So: our function $f(z) - z^{-2}$ is holomorphic near 0.

- But $f(z)$ is also supposed to be an elliptic function, so $f(z)$ also has a double pole at each point ω of the lattice Λ.
- So by the same exact argument, $f(z) - (z - \omega)^{-2}$ will be holomorphic near an arbitrary $\omega \in \Lambda$.
- So now, we ask: what happens if we subtract all of these "pole contributions" $(z - \omega)^{-2}$ for all $\omega \in \Lambda$ from $f(z)$?

Constructing Elliptic Functions, III

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- So by the same exact argument, $f(z) - (z-\omega)^{-2}$ will be holomorphic near an arbitrary $\omega \in \Lambda$.
- So now, we ask: what happens if we subtract all of these "pole contributions" $(z-\omega)^{-2}$ for all $\omega \in \Lambda$ from $f(z)$?
- The resulting function would then have no poles at all, hence be entire, hence (under the assumption it is elliptic) constant.
- By shifting so that this constant is zero, we would obtain a formula for $f(z)$: namely, $f(z) = \sum_{\omega \in \Lambda} (z-\omega)^{-2}$.

Constructing Elliptic Functions, IV

Okay, so now we've constructed an elliptic function:
$f(z) = \sum_{\omega \in \Lambda} (z - \omega)^{-2}$.... right?

Constructing Elliptic Functions, IV

Okay, so now we've constructed an elliptic function:
$f(z) = \sum_{\omega \in \Lambda} (z - \omega)^{-2}$.... right?

- Well... no, not so much.
- Unfortunately, there's a critical problem: the series $\sum_{\omega \in \Lambda} (z - \omega)^{-2}$ does not converge absolutely!
- This is bad, because if we take a non-absolutely-convergent series, we cannot manipulate it in the ways we'd like.

Constructing Elliptic Functions, V

Exercise: Let $\omega = a\omega_1 + b\omega_2$. Show that $|\omega|^2 = xa^2 + yab + zb^2$ is a positive-definite quadratic form in (a, b), where $x = |\omega_1|^2$, $y = 2\mathrm{Re}(\omega_1\overline{\omega_2})$, $z = |\omega_2|^2$.

Constructing Elliptic Functions, V

<u>Exercise</u>: Let $\omega = a\omega_1 + b\omega_2$. Show that $|\omega|^2 = xa^2 + yab + zb^2$ is a positive-definite quadratic form in (a, b), where $x = |\omega_1|^2$, $y = 2\mathrm{Re}(\omega_1\overline{\omega_2})$, $z = |\omega_2|^2$.

<u>Exercise</u>: Show that if $Q(a, b)$ is a positive-definite real quadratic form, then $\sum_{(0,0)\neq(a,b)\in\mathbb{Z}\times\mathbb{Z}} \dfrac{1}{Q(a, b)^k}$ diverges for $k \leq 1$ and converges absolutely for $k > 1$. [Hint: Compare to the corresponding integral, diagonalize the quadratic form, and use polar coordinates.]

Constructing Elliptic Functions, V

Exercise: Let $\omega = a\omega_1 + b\omega_2$. Show that $|\omega|^2 = xa^2 + yab + zb^2$ is a positive-definite quadratic form in (a, b), where $x = |\omega_1|^2$, $y = 2\mathrm{Re}(\omega_1\overline{\omega_2})$, $z = |\omega_2|^2$.

Exercise: Show that if $Q(a, b)$ is a positive-definite real quadratic form, then $\sum_{(0,0)\neq(a,b)\in\mathbb{Z}\times\mathbb{Z}} \dfrac{1}{Q(a,b)^k}$ diverges for $k \leq 1$ and converges absolutely for $k > 1$. [Hint: Compare to the corresponding integral, diagonalize the quadratic form, and use polar coordinates.]

Exercise: Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice. Show that $\sum_{0\neq\omega\in\Lambda} |\omega|^{-k}$ diverges for $k \leq 2$ and converges absolutely for $k > 2$.

Constructing Elliptic Functions, VI

Let $\sum_{\omega \in \Lambda *}$ denote a sum over nonzero elements in Λ, for z bounded (e.g., in a fundamental region).

- Then the absolute value series is $\sum_{\omega \in \Lambda} |z - \omega|^{-2} = \frac{1}{z^2} + \sum_{\omega \in \Lambda *} \left| \frac{1}{\omega^2} + \frac{2z}{\omega^3} + \frac{3z^2}{\omega^4} + \cdots \right| = \sum_{\omega \in \Lambda *} \left| \omega^{-2} + O(\omega^{-3}) \right|$ is on the order of $\sum_{\omega \in \Lambda *} |\omega|^{-2}$ which diverges by the exercises on the last slide.

Constructing Elliptic Functions, VI

Let $\sum_{\omega \in \Lambda *}$ denote a sum over nonzero elements in Λ, for z bounded (e.g., in a fundamental region).

- Then the absolute value series is $\sum_{\omega \in \Lambda} |z - \omega|^{-2} = \frac{1}{z^2} + \sum_{\omega \in \Lambda *} \left| \frac{1}{\omega^2} + \frac{2z}{\omega^3} + \frac{3z^2}{\omega^4} + \cdots \right| =$ $\sum_{\omega \in \Lambda *} \left| \omega^{-2} + O(\omega^{-3}) \right|$ is on the order of $\sum_{\omega \in \Lambda *} |\omega|^{-2}$ which diverges by the exercises on the last slide.
- Notice, however, that this series just barely fails to converge: indeed, if we could get rid of the ω^{-2} term, then the remaining series would be $\sum_{\omega \in \Lambda *} \left| \frac{2z}{\omega^3} + \frac{3z^2}{\omega^4} + \cdots \right| = 2|z| \sum_{\omega \in \Lambda *} (|\omega|^{-3} + O(|\omega|^{-4}))$, which does converge absolutely.

How Do We Solve A Problem Like ω^{-2}?

So how can we remove that ω^{-2} term?

- Dumb idea: just subtract ω^{-2} from each term of the series where $\omega \neq 0$.
- In other words, use instead
$$f(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda *} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$
- In fact, this is actually a rather good idea, because by the calculations we just did this series does converge absolutely and uniformly on compact subsets of $\mathbb{C}$ to a meromorphic function having a double pole at each element of Λ.
- But now it's not so clear that this is actually an elliptic function, because when we shift the series by $\omega \in \Lambda$, its value is not obviously the same anymore.

The Weierstrass $\wp$-Function, I

So now let's define some things:

Definition

Let ω_1, ω_2 are $\mathbb{R}$-linearly independent complex numbers and $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the associated complex lattice.

The Weierstrass $\wp$-function (with respect to Λ) is defined to be

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

The Eisenstein series of weight $2k$ (with respect to Λ) is

$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda*} \frac{1}{\omega^{2k}}$ *where the sums are over all nonzero $\omega \in \Lambda$.*

- When Λ is clear from context, we will just write $\wp(z)$ in place of $\wp(z; \Lambda)$ and G_{2k} in place of $G_{2k}(\Lambda)$.
- We index as G_{2k} because the G_{2k-1} are all zero.

The Weierstrass $\wp$-Function, II

Theorem (Properties of $\wp$ and G_{2k}, Part 1)

Let Λ be a complex lattice with

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda*}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \text{ and } G_{2k}(\Lambda) = \sum_{\omega\in\Lambda*}\frac{1}{\omega^{2k}}$$

1. *The Eisenstein series $G_{2k}(\Lambda)$ is absolutely convergent for $k > 1$ but not for $k \leq 1$.*
2. *The series defining $\wp(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$.*
3. *The $\wp$-function is meromorphic on $\mathbb{C}$ with a double pole with residue 0 at each point of Λ (and no other poles).*
4. *The $\wp$-function is an even function: $\wp(-z) = \wp(z)$.*

The Weierstrass $\wp$-Function, III

1. The Eisenstein series $G_{2k}(\Lambda) = \sum_{\omega \in \Lambda *} \frac{1}{\omega^{2k}}$ is absolutely convergent for $k > 1$ but not for $k \leq 1$.

Proof:

The Weierstrass $\wp$-Function, III

1. The Eisenstein series $G_{2k}(\Lambda) = \sum_{\omega \in \Lambda*} \frac{1}{\omega^{2k}}$ is absolutely convergent for $k > 1$ but not for $k \leq 1$.

Proof:

- By standard geometric results about lattices, if the fundamental parallelogram for Λ has area Δ, then the number of $\omega \in \Lambda$ with $|\omega| \leq R$ is $\frac{\pi}{\Delta} R^2 + O(R)$ as $R \to \infty$.
- Then for arbitrary R and sufficiently large d, the number n_R of $\omega \in \Lambda$ with $R \leq |\omega| < R + d$ is $\Theta(R)$.
- Hence by grouping ω together into the annuli $R \leq |\omega| < R + d$, by the comparison test we see that $\sum_{\omega \in \Lambda^*} |\omega|^{-2k}$ has the same behavior as the series $\sum_{R=1}^{\infty} \frac{\#\{\omega \in \Lambda : Rd \leq |\omega| < Rd + d\}}{(Rd)^k} \sim \sum_{R=1}^{\infty} \frac{R}{R^{2k}}$ which as a p-series is convergent for $k > 1$ and divergent for $k \leq 1$.

The Weierstrass $\wp$-Function, IV

2. The series $\wp(z;\Lambda) = \dfrac{1}{z^2} + \sum_{\omega\in\Lambda*}\left[\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2}\right]$ converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$.

Proof:

The Weierstrass $\wp$-Function, IV

2. The series $\wp(z;\Lambda) = \dfrac{1}{z^2} + \sum_{\omega\in\Lambda*}\left[\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2}\right]$ converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$.

Proof:

- For $|\omega| > 2|z|$, we have
  $$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \frac{|z|\,|2\omega - z|}{|\omega|^2\,|\omega - z|^2} \le \frac{10\,|z|}{|\omega|^3}.$$
- Hence the tail of the series $\dfrac{1}{z^2} + \sum_{\omega\in\Lambda*}\left[\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2}\right]$ with $|\omega| > 2|z|$ is bounded in absolute value by $\sum_{\omega\in\Lambda^*}\dfrac{10\,|z|}{|\omega|^3}$ which converges absolutely by (a).
- Hence by the Weierstrass M-test, the series defining $\wp(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$.

The Weierstrass $\wp$-Function, V

3. The $\wp$-function is meromorphic on $\mathbb{C}$ with a double pole with residue 0 at each point of Λ (and no other poles).

Proof:

The Weierstrass $\wp$-Function, V

3. The $\wp$-function is meromorphic on $\mathbb{C}$ with a double pole with residue 0 at each point of Λ (and no other poles).

Proof:

- For $\omega \in \Lambda$ the local expansion of $\wp(z)$ at ω is $(z-\omega)^2 + O((z-\omega)^0)$ so there is a double pole with residue 0 at Λ.
- Since the series for $\wp$ is absolutely convergent on $\mathbb{C}\backslash\Lambda$ by (2), $\wp$ has no other poles.

The Weierstrass $\wp$-Function, VI

4. The $\wp$-function is an even function: $\wp(-z) = \wp(z)$.

Proof:

The Weierstrass $\wp$-Function, VI

4. The $\wp$-function is an even function: $\wp(-z) = \wp(z)$.

Proof:

- We have $\wp(-z)$
$$= \frac{1}{(-z)^2} + \sum_{\omega \in \Lambda*} \left[\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2} \right]$$
$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$
$$= \wp(z)$$
by substituting $\omega \mapsto -\omega$ in the sum.

The Weierstrass $\wp$-Function, VII

Theorem (Properties of $\wp$ and G_{2k}, Part 2)

Let Λ be a complex lattice with

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda*}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \text{ and } G_{2k}(\Lambda) = \sum_{\omega\in\Lambda*}\frac{1}{\omega^{2k}}$$

5. *The derivative $\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}$ is an odd function with a triple pole at each point of Λ (and no other poles).*
6. *The $\wp$-function and its derivative are elliptic functions with respect to Λ.*
7. *The field of even elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z))$.*
8. *The field of elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z), \wp'(z))$.*

The Weierstrass $\wp$-Function, VI

5. The derivative $\wp'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}$ is an odd function with a triple pole at each point of Λ (and no other poles).

Proof:

The Weierstrass $\wp$-Function, VI

5. The derivative $\wp'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}$ is an odd function with a triple pole at each point of Λ (and no other poles).

Proof:

- Since the series for $\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda *} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$ converges uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$ its derivative is obtained by differentiating the series term by term, immediately yielding the given sum.
- Then $\wp'$ is odd since derivatives of even functions are odd, and $\wp'$ has a triple pole at each point of Λ since differentiating a pole creates a pole of one higher order but does not otherwise create new poles.

The Weierstrass $\wp$-Function, V

6. The $\wp$-function and its derivative are elliptic functions with respect to Λ.

Proof:

The Weierstrass $\wp$-Function, V

6. The $\wp$-function and its derivative are elliptic functions with respect to Λ.

Proof:

- First, $\wp'(z)$ is elliptic since the series expression in (5) is clearly invariant under translation by elements of Λ.
- For $\wp(z)$, taking the antiderivative of $\wp'(z+\omega) = \wp'(z)$ yields $\wp(z+\omega) = \wp(z) + C_\omega$ for some constant C_ω depending only on ω and not on z.
- Setting $z = -\omega/2$ and using evenness of $\wp$ immediately yields $C_\omega = 0$, and so $\wp$ is also elliptic.

The Weierstrass $\wp$-Function, IV

7. The field of even elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z))$.

Discussion:

The Weierstrass $\wp$-Function, IV

7. The field of even elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z))$.

Discussion:

- Suppose that f is an even elliptic function, with $f(-z) = f(z) = f(z+\omega)$ for all $\omega \in \Lambda$.
- Our goal is to construct an elliptic function having the same zeroes and poles as f using only expressions of the form $\wp(z) - c$ for constants c: then the ratio of f to this function is elliptic and has no zeroes nor poles hence is constant.
- Let D be a fundamental parallelogram for Λ and let H be a fundamental domain for $(\mathbb{C}/\Lambda)/\{\pm 1\}$ (i.e., half of the fundamental parallelogram, consisting of a unique representative chosen among the two points $\{\zeta, \omega_1 + \omega_2 - \zeta\}$ for each $\zeta \in D$).

The Weierstrass $\wp$-Function, IV

7. The field of even elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z))$.

Proof (part 1):

- Now, since f is even, for each $\zeta \in D$ we have $\operatorname{ord}_{\zeta}(f) = \operatorname{ord}_{\omega_1+\omega_2-\zeta}(f)$, and also for the half-lattice points ζ with $2\zeta \in \Lambda$, we see that $\operatorname{ord}_{\zeta}(f)$ is even because $f^{(i)}(z) = (-1)^{i-1} f^{(i)}(-z)$ hence $f^{(i)}(\zeta) = 0$ since $\zeta \equiv -\zeta$ mod Λ.
- Now list all of the zeroes $\{a_1, \ldots, a_k\}$ and poles $\{b_1, \ldots, b_k\}$ of f inside H, including appropriate multiplicities, where we list any zero or pole ζ with $2\zeta \in \Lambda$ with half multiplicity.
- We claim that the function $g(z) = \prod_{i=1}^{k} \dfrac{\wp(z) - \wp(a_i)}{\wp(z) - \wp(b_i)}$ has the same zero and pole orders as f.

The Weierstrass $\wp$-Function, III

7. The field of even elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z))$.

Proof (part 2):

- We claim that the function $g(z) = \prod_{i=1}^{k} \dfrac{\wp(z) - \wp(a_i)}{\wp(z) - \wp(b_i)}$ has the same zero and pole orders as f.
- To see this, observe that $\wp(z) - \wp(a_i)$ has a zero at a_i and a zero at $-a_i$ (if $a_i = -a_i$ this is a double zero) and a double pole at 0.
- Hence by construction, $g(z)$ has the same zero and pole order as f does at all points except possibly at 0.
- But because f and g are both elliptic, the sum of both of their orders over all points is 0, and so they must have the same order at 0 as well. Hence the ratio $f(z)/g(z)$ is elliptic with no zeroes or poles, so it is constant. We conclude that $f(z) \in \mathbb{C}(\wp(z))$ as claimed.

The Weierstrass $\wp$-Function, II

7. The field of elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z), \wp'(z))$.

Proof:

The Weierstrass $\wp$-Function, II

7. The field of elliptic functions $\mathbb{C}(\Lambda)$ is equal to $\mathbb{C}(\wp(z), \wp'(z))$.

Proof:

- If $f(z)$ is elliptic, then both of the functions $\dfrac{f(z)+f(-z)}{2}$ and $\dfrac{f(z)-f(-z)}{2\wp'(z)}$ are even and elliptic, hence by (7) they are both rational functions of $\wp(z)$.
- Then if $g(\wp(z)) = \dfrac{f(z)+f(-z)}{2}$, $h(\wp(z)) = \dfrac{f(z)-f(-z)}{2\wp'(z)}$, we have $f(z) = g(\wp(z)) + \wp'(z) \cdot h(\wp(z)) \in \mathbb{C}(\wp(z), \wp'(z))$.

In fact, this shows every elliptic function is a rational function in $\wp(z)$ plus $\wp'(z)$ times another rational function in $\wp(z)$.

The Weierstrass $\wp$-Function, I

The goal of this entire construction was to find the analogues of the coordinate functions x and y on $\mathbb{C}/\Lambda$.

- Since $\wp(z)$ has a double pole at 0 and $\wp'(z)$ has a triple pole at 0, these two functions are natural candidates for x and y, following the Riemann-Roch analogy (in which x was constructed as an element of $L(2P)$ not in $L(P)$ and y was constructed as an element of $L(3P)$ not in $L(2P)$).
- We therefore can hope that there exists a relation of the form $\wp'(z)^2 = \wp(z)^3 + A\wp(z) + B$ for some constants A and B (which necessarily will depend on the lattice).

The Weierstrass $\wp$-Function, 0

Indeed, we know there must be some algebraic relation between $\wp(z)$ and $\wp'(z)$, because $\wp'(z)^2$ is an even elliptic function, hence by (7) in the proposition above it must be a rational function of $\wp(z)$.

- We can use (7) to compute the precise relation, which requires only understanding the zeroes and poles of $\wp'(z)$. This will give us one form of the cubic expression we seek.
- Alternatively, we could simply calculate the Laurent expansions of each of the terms near $z = 0$ and compute an appropriate linear combination that is holomorphic: then it will be a holomorphic elliptic function hence constant. This will give us a second form of the cubic expression.