1. Decide whether each residue class has a multiplicative inverse modulo m. If so, find it, and if not, explain why not:
   (a) $\overline{10}$ mod 25.
   (b) $\overline{11}$ mod 25.
   (c) $\overline{12}$ mod 25.
   (d) $\overline{30}$ mod 42.
   (e) $\overline{31}$ mod 42.
   (f) $\overline{32}$ mod 42.

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2. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) irreflexive, (vi) an equivalence relation, (vii) a partial ordering, and (viii) a total ordering.
   (a) $R = \{(1, 1), (2, 1), (2, 2)\}$ on the set $\{1, 2\}$.
   (b) $R = \{(1, 2), (2, 1)\}$ on the set $\{1, 2\}$.
   (c) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ on the set $\{1, 2, 3, 4\}$.
   (d) The divisibility relation on the set $\{2, -3, 4, -5, 6\}$.
   (e) The divisibility relation on the set $\{2, -4, -12, 36\}$.
   (f) The relation R on $\mathbb{Z}$ with $a\ R\ b$ precisely when $|a| \equiv |b|$ modulo 5.
   (g) The relation R on $\mathbb{R}$ with $a\ R\ b$ precisely when $ab > 0$.

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3. For each of the following functions $f : A \to B$, determine whether (i) f is one-to-one, (ii) f is onto, (iii) f is a bijection.
   (a) $f = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ from $\{1, 2, 3, 4\}$ to itself.
   (b) $f = \{(1, 3), (2, 4), (3, 1), (4, 4)\}$ from $\{1, 2, 3, 4\}$ to itself.
   (c) $f(x) = 2x$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
   (d) $f(n) = 2n$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.
   (e) $f(x) = \dfrac{x}{x-1}$ from $A = \mathbb{R}\backslash\{1\}$ to $B = \mathbb{R}$.
   (f) $f(x) = x^3$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.

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4. Identify the ordered pairs in the equivalence relation that corresponds to the partition $\{1, 2, 4\}, \{3, 5\}, \{6\}$ of $\{1, 2, 3, 4, 5, 6\}$.

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5. Show $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| = |y|\}$ is an equivalence relation on $\mathbb{Z}$ and list the equivalence classes of 0, 1, 2, −2, 4.

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6. Find all minimal, maximal, largest, and smallest elements of the set $\{-2, -1, 1, 2, 3, 6\}$ under the divisibility ordering, and on the ordering $a\,R\,b$ when $a = b$ or $a + 2 < b$.

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7. Suppose $f : A \to B$ is a function.
   (a) If f is one-to-one, show that there is a bijection between A and $\text{im}(f)$. Deduce that $\#A = \#\text{im}(f)$.
   (b) If A and B are both finite and $\#A = \#B$, show that f is one-to-one implies that f is onto.

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8. Let A, B, C be sets, R be a relation, f, g be functions, and n be a positive integer. Prove the following:

(a) Prove that the sum of any six consecutive integers is congruent to 3 modulo 6.

(b) Show that the only equivalence relation R on A that is a function from A to A is the identity relation.

(c) Prove that $5^n + 6^n \equiv 0 \pmod{11}$ if and only if n is odd.

(d) Prove that if A is countable and B is uncountable, then $B \backslash A$ is uncountable.

(e) Suppose that $f : A \to A$ is a function. Show that $f(f(a)) = a$ for all $a \in A$ if and only if f^{-1} exists and $f^{-1}(a) = f(a)$ for all $a \in A$.

(f) Suppose A, B are fixed sets and let T be the collection of all sets D with $D \subseteq A$ and $D \subseteq B$. Show that $A \cap B$ is the largest element of T.

(g) Prove that the set $\mathbb{Q} \times \mathbb{Z}$ is countable and that the set $\mathbb{R} \times \mathbb{Z}$ is uncountable.

(h) Prove that $7^n + 5$ is divisible by 6 for all positive integers n.

(i) Suppose $f : B \to C$ is one-to-one. If $g, h : A \to B$ have $f \circ g = f \circ h$, show that $g = h$.

(j) Suppose R is a reflexive and transitive relation on a set A. Show that $S = R \cap R^{-1}$ is an equivalence relation on A.

(k) If n is any positive integer, prove that $n - 1$ is invertible modulo n and its multiplicative inverse is itself.

(l) Prove that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$, the set of rational numbers strictly between 0 and 1.

(m) Prove that if $f : A \to B$ is one-to-one and $S \subseteq A$, then $f^{-1}(f(S)) = S$.

(n) Prove that if $f : A \to B$ is onto and $T \subseteq B$, then $f(f^{-1}(T)) = T$.

(o) Suppose $f : A \to B$ is a bijection. Show that $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ given by $\tilde{f}(S) = \{f(s) : s \in S\}$ is also a bijection.

(p) If $R, S : A \to B$ are relations, prove that $R^{-1} \cap S^{-1} = (R \cap S)^{-1}$.

(q) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Show that $a(b + c) \equiv b(a + d) \pmod{n}$.

(r) Suppose R is an equivalence relation on A and S is an equivalence relation on B. Show that the relation T on $A \times B$ given by $(a, b)\, T\, (a', b')$ precisely when $a\, R\, a'$ and $b\, S\, b'$ is an equivalence relation.

(s) Suppose $\mathcal{P}$ is a partition of A and $\mathcal{Q}$ is a partition of B. Show that $\mathcal{P} \times \mathcal{Q} = \{X \times Y : X \in \mathcal{P}, Y \in \mathcal{Q}\}$ is a partition of $A \times B$.

(t) Let S be the set of equivalence classes of an equivalence relation R on A and define the function $f : A \to S$ via $f(a) = [a]$. Show that f is one-to-one if and only if R is the identity relation.

(u) Prove that the set of all finite subsets of $\mathbb{Z}$ is countable.
