Math 1365 (Intensive Mathematical Reasoning)

Lecture #25 of 35 ∼ November 6, 2023

Inverse Functions

- Inverse Functions
- Properties of Inverse Functions
- Bijections

This material represents §3.4.3 + §3.4.4 from the course notes.

Announcement: I will be giving Wednesday's lecture in person, in the regular classroom.

Recall, I

Recall some definitions:

- A <u>function</u> $f : A \to B$ from a domain A to a target B is a subset of $A \times B$ such that for every $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$, and in that case we write $f(a) = b$.
- The <u>identity function</u> $i_A : A \to A$ has $i_A(a) = a$ for all $a \in A$.

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- The <u>identity function</u> $i_A : A \to A$ has $i_A(a) = a$ for all $a \in A$.

- The function $f : A \to B$ is <u>one-to-one</u> (<u>injective</u>) when for any $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

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- The <u>identity function</u> $i_A : A \to A$ has $i_A(a) = a$ for all $a \in A$.

- The function $f : A \to B$ is <u>one-to-one</u> (<u>injective</u>) when for any $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

- The <u>image</u> $\mathrm{im}(f) = \{b \in B : \exists a \in A \text{ with } f(a) = b\}$ is the set of elements $b \in B$ of the form $f(a)$ for some $a \in A$.
- The function $f : A \to B$ is <u>onto</u> (<u>surjective</u>) when $\mathrm{im}(f) = B$. Explicitly: for any $b \in B$, there exists an $a \in A$ with $f(a) = b$.

Recall, II

Recall also some properties of function composition:

Definition

For functions $g : A \to B$ and $f : B \to C$, their composite function $f \circ g : A \to C$ is defined via $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

Remember that function composition is applied right-to-left: in the composition $f \circ g$, the function g is the one that is applied first.

Proposition (Properties of Composition)

Suppose A, B, C, D are sets.

1. *Function composition is associative: If $f : C \to D$, $g : B \to C$, and $h : A \to B$ are any functions then $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as functions from A to D.*
2. *The identity function behaves as a left and right identity: For any $f : A \to B$, $f \circ i_A = f$ and $i_B \circ f = f$.*

Inverses of Functions, I

Our goal now is to discuss when a function $f : A \to B$ has an inverse function $f^{-1} : B \to A$.

- The idea is that we would like the inverse function f^{-1} to "undo" the action of f: if $f(a) = b$, then $f^{-1}(b) = a$.
- On the level of ordered pairs, we want $(a, b) \in f$ when $(b, a) \in f^{-1}$, meaning that f^{-1} is the inverse relation to f.
- So the question boils down to asking when the inverse relation f^{-1} is a function from B to A.

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- On the level of ordered pairs, we want $(a, b) \in f$ when $(b, a) \in f^{-1}$, meaning that f^{-1} is the inverse relation to f.
- So the question boils down to asking when the inverse relation f^{-1} is a function from B to A.
- We saw in the examples last lecture that when f is one-to-one, then f^{-1} seems to be a function from $\mathrm{im}(f)$ to A.
- If we then add the additional condition that f be onto, then f^{-1} will be a function from B back to A.

So now, let's prove these things.

Inverses of Functions, II

We can now establish the precise relationship between being one-to-one (or onto) and the existence of an inverse function. First, some preliminary pieces:

Proposition (One-to-One, Onto, and Inverses)

Suppose $f : A \to B$ is a function.

1. *The inverse relation f^{-1} is a function (from $\mathrm{im}(f)$ to A) if and only if f is one-to-one.*
2. *If $f^{-1} : B \to A$ is a function, then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.*
3. *If there exists a function $g : B \to A$ such that $g \circ f = i_A$, then f is one-to-one.*
4. *If there exists a function $g : B \to A$ such that $f \circ g = i_B$, then f is onto.*

Inverses of Functions, III

1. For $f : A \to B$, the inverse relation f^{-1} is a function (from $\mathrm{im}(f)$ to A) if and only if f is one-to-one.

Example:

- Consider the function $f : \{1, 2, 3\} \to \{5, 6, 7, 8\}$ with $f = \{(1, 5), (2, 7), (3, 8)\}$, so $f(1) = 5$, $f(2) = 7$, $f(3) = 8$.

Inverses of Functions, III

1. For $f : A \to B$, the inverse relation f^{-1} is a function (from $\mathrm{im}(f)$ to A) if and only if f is one-to-one.

Example:

- Consider the function $f : \{1, 2, 3\} \to \{5, 6, 7, 8\}$ with $f = \{(1, 5), (2, 7), (3, 8)\}$, so $f(1) = 5$, $f(2) = 7$, $f(3) = 8$.
- Then f is one-to-one since it sends the 3 elements in the domain to different places in the target.
- We see that $f^{-1} = \{(5, 1), (7, 2), (8, 3)\}$, and this is indeed a function from the set $\{5, 7, 8\} = \mathrm{im}(f)$ to the set $\{1, 2, 3\}$ which is the domain of f.

Inverses of Functions, IV

1. The inverse relation f^{-1} is a function (from $\mathrm{im}(f)$ to A) if and only if f is one-to-one.

<u>Proof</u>:

Inverses of Functions, IV

1. The inverse relation f^{-1} is a function (from $\mathrm{im}(f)$ to A) if and only if f is one-to-one.

Proof:

- Note that f^{-1} is a function precisely when $(c, a) \in f^{-1}$ and $(c, b) \in f^{-1}$ implies $a = b$.
- This condition is equivalent to saying that if $(a, c) \in f$ and $(b, c) \in f$ then $a = b$.
- That is equivalent to saying that if $f(a) = c = f(b)$ then $a = b$.
- But this last condition is precisely the same as saying f is one-to-one.

Inverses of Functions, IV

2. If $f^{-1} : B \to A$ is a function, then $f^{-1} \circ f = i_A$, the identity function on A, and $f \circ f^{-1} = i_B$, the identity function on B.

Example:

- Consider the function $f : \{1, 2, 3\} \to \{5, 7, 8\}$ with $f = \{(1, 5), (2, 7), (3, 8)\}$, so $f(1) = 5$, $f(2) = 7$, $f(3) = 8$.

Inverses of Functions, IV

2. If $f^{-1} : B \to A$ is a function, then $f^{-1} \circ f = i_A$, the identity function on A, and $f \circ f^{-1} = i_B$, the identity function on B.

Example:

- Consider the function $f : \{1, 2, 3\} \to \{5, 7, 8\}$ with $f = \{(1, 5), (2, 7), (3, 8)\}$, so $f(1) = 5$, $f(2) = 7$, $f(3) = 8$.
- Then $f^{-1} = \{(5, 1), (7, 2), (8, 3)\}$ is a function from $\{5, 7, 8\}$ to $\{1, 2, 3\}$, with $f^{-1}(5) = 1$, $f^{-1}(7) = 2$, $f^{-1}(8) = 3$.
- Then $f^{-1} \circ f$ is a function from $\{1, 2, 3\}$ to $\{1, 2, 3\}$, and
  $(f^{-1} \circ f)(1) = f^{-1}(f(1)) = f^{-1}(5) = 1$,
  $(f^{-1} \circ f)(2) = f^{-1}(f(2)) = f^{-1}(7) = 2$, and
  $(f^{-1} \circ f)(3) = f^{-1}(f(3)) = f^{-1}(8) = 3$.
- So we see that $f^{-1} \circ f$ is just the identity function on $\{1, 2, 3\}$ since it maps each element to itself.
- In the same way we can see $f \circ f^{-1}$ is the identity on $\{5, 7, 8\}$.

Inverses of Functions, V

2. If $f^{-1} : B \to A$ is a function, then $f^{-1} \circ f = i_A$, the identity function on A, and $f \circ f^{-1} = i_B$, the identity function on B.

<u>Proof</u>:

Inverses of Functions, V

2. If $f^{-1} : B \to A$ is a function, then $f^{-1} \circ f = i_A$, the identity function on A, and $f \circ f^{-1} = i_B$, the identity function on B.

Proof:

- By definition of the identity function, the statement $f^{-1} \circ f = i_A$ is the same as saying that $f^{-1}(f(a)) = a$ for all $a \in A$, and the statement $f \circ f^{-1} = i_B$ is the same as saying that $f(f^{-1}(b)) = b$ for all $b \in B$.
- First, note that $f^{-1} \circ f$ is a function from A to A.
- Now let $a \in A$ be arbitrary and set $b = f(a) \in B$. Then $(a, b) \in f$ so $(b, a) \in f^{-1}$, meaning that $f^{-1}(b) = a$.
- Then $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$ by the above.
- But since a was arbitrary, and $f^{-1} \circ f$ and i_A have the same domain and target and take the same values for all $a \in A$, they are equal as functions.
- A similar argument works to show $f \circ f^{-1} = i_B$.

Inverses of Functions, VI

3. If there exists a function $g : B \to A$ such that $g \circ f = i_A$, then f is one-to-one.

Proof:

Inverses of Functions, VI

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Proof:

- To show this we need to establish that if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
- So suppose $g : B \to A$ has $g \circ f = i_A$ and that $f(a_1) = f(a_2)$.
- Applying g to both sides yields $g(f(a_1)) = g(f(a_2))$.
- But we have $g(f(a_1)) = (g \circ f)(a_1) = i_A(a_1) = a_1$, and in the same way, $g(f(a_2)) = (g \circ f)(a_2) = i_A(a_2) = a_2$.
- Thus we see $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$, so f is one-to-one.

Inverses of Functions, VII

4. If there exists a function $g : B \to A$ such that $f \circ g = i_B$, then f is onto.

Proof:

Inverses of Functions, VII

4. If there exists a function $g : B \to A$ such that $f \circ g = i_B$, then f is onto.

Proof:

- To show this we need to establish that for each $b \in B$ there exists $a \in A$ with $f(a) = b$.
- So suppose $g : B \to A$ has $f \circ g = i_B$ and let $b \in B$ be arbitrary.
- Then we have $b = i_B(b) = (f \circ g)(b) = f(g(b))$.
- This means if we set $a = g(b)$, then $f(a) = b$: thus f is onto.

Inverses of Functions, VIII

By combining the results we just proved, we can give several equivalent characterizations of when a function has an inverse function:

Theorem (Inverse Functions)

Suppose $f : A \to B$ is a function. Then the following are equivalent:

1. *f is one-to-one and onto.*
2. *f^{-1} is a function from B to A.*
3. *There exists a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.*

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We show that (1) implies (2), that (2) implies (3), and that (3) implies (1). This is sufficient because the other implications, such as (1) implies (3), follow from these three because logical implication is transitive as we have previously noted.

Inverses of Functions, IX

If $f : A \to B$ is a function, the following are equivalent:

1. f is one-to-one and onto.
2. f^{-1} is a function from B to A.
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Proof:

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2. f^{-1} is a function from B to A.
3. There exists $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

Proof:

- (1) $\Rightarrow$ (2): If f is one-to-one, then f^{-1} is a function from $\mathrm{im}(f)$ to A by result (1) earlier. If f is also onto, then $\mathrm{im}(f) = B$, and so f^{-1} is a function from B to A.

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Proof:

- (1) $\Rightarrow$ (2): If f is one-to-one, then f^{-1} is a function from $\mathrm{im}(f)$ to A by result (1) earlier. If f is also onto, then $\mathrm{im}(f) = B$, and so f^{-1} is a function from B to A.
- (2) $\Rightarrow$ (3): If f^{-1} is a function from B to A, then simply take $g = f^{-1}$; by result (2) earlier, $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

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Proof:

- (1) $\Rightarrow$ (2): If f is one-to-one, then f^{-1} is a function from $\mathrm{im}(f)$ to A by result (1) earlier. If f is also onto, then $\mathrm{im}(f) = B$, and so f^{-1} is a function from B to A.
- (2) $\Rightarrow$ (3): If f^{-1} is a function from B to A, then simply take $g = f^{-1}$; by result (2) earlier, $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.
- (3) $\Rightarrow$ (1): If there exists a function $g : B \to A$ such that $g \circ f = i_A$ then by result (3) earlier, we see f is one-to-one. Since g also has the property that $f \circ g = i_B$, then by result (4) earlier, we see f is also onto.

Inverses of Functions, X

We can also deduce that (when it exists) the inverse function is the unique two-sided inverse of f:

Corollary (Uniqueness of Inverse)

Suppose $f : A \to B$ and $g : B \to A$ are functions such that $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

Proof:

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Suppose $f : A \to B$ and $g : B \to A$ are functions such that $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

Proof:

- If there exists such a function g, then by the theorem above, f^{-1} is a function from B to A and it satisfies the same properties as g.
- Then by the basic properties of function composition, we have $g = i_A \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ i_B = f^{-1}$, as required.

Inverses of Functions, XI

So now that we have characterized when $f : A \to B$ has an inverse function $f^{-1} : B \to A$ – namely, when f is both one-to-one and onto – how do we calculate the inverse itself?

- When f is described as a list of ordered pairs, as we have already explained, f^{-1} is obtained simply by reversing all of the pairs.
- When f is described as a rule (typically, for functions written algebraically), to find the inverse we simply solve the equation $y = f(x)$ for x in terms of y: this will give $x = f^{-1}(y)$.

Inverses of Functions, XII

Example: Verify that the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = 3x - 2$ is invertible and find its inverse function.

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- To show that h is one-to-one, notice that $h(a) = h(b)$ is the same as $3a - 2 = 3b - 2$, and this can easily be rearranged to obtain $a = b$.
- To find h^{-1}, we solve $y = 3x - 2$ for x in terms of y.
- We obtain $x = \dfrac{y+2}{3}$, so $h^{-1}(y) = \dfrac{y+2}{3}$. Note that this calculation also shows that h is onto.

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- We obtain $x = \dfrac{y+2}{3}$, so $h^{-1}(y) = \dfrac{y+2}{3}$. Note that this calculation also shows that h is onto.

Notice here that h is a composite function: h scales its argument by 3 and then subtracts 2.

- Its inverse function reverses each of these operations in the opposite order: namely, h^{-1} first adds 2 and then divides its argument by 3.

Inverses of Functions, XIII

The observation in this example holds in general:

Proposition (Properties of Inverses)

Suppose that $f : B \to C$ and $g : A \to B$ are invertible functions (i.e., one-to-one and onto). Then

1. *The function $f^{-1} : C \to B$ is invertible, with inverse $(f^{-1})^{-1} = f$.*
2. *The function $f \circ g : A \to C$ is invertible, with inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

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2. *The function $f \circ g : A \to C$ is invertible, with inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

By our theorem on invertible functions, to show two functions are inverses we need only verify that composing them in either order yields the appropriate identity function.

Inverses of Functions, XIV

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Proof:

Inverses of Functions, XIV

1. The function $f^{-1} : C \to B$ is invertible, with inverse $(f^{-1})^{-1} = f$.

Proof:

- As we have shown, we have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.
- But this means f fills the role of the inverse function $(f^{-1})^{-1}$.
- But since the inverse of a function is unique, that means f^{-1} is invertible and its inverse is f, as claimed.

Inverses of Functions, XV

2. The function $f \circ g : A \to C$ is invertible, with inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Proof:

Inverses of Functions, XV

2. The function $f \circ g : A \to C$ is invertible, with inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Proof:

- First observe that $[f \circ g] \circ [g^{-1} \circ f^{-1}] = f \circ [g \circ g^{-1}] \circ f^{-1} = f \circ i_B \circ f^{-1} = f \circ f^{-1} = i_C$.
- Likewise, $[g^{-1} \circ f^{-1}] \circ [f \circ g] = g^{-1} \circ [f^{-1} \circ f] \circ g = g^{-1} \circ i_B \circ g = g^{-1} \circ g = i_A$.
- Hence since the composition yields the needed identity function in both orders, that means $f \circ g$ is invertible and its inverse is $g^{-1} \circ f^{-1}$, as claimed.

Inverses of Functions, XVI

Example: Verify that the function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $h(x, y) = (x - y, 3y + 1)$ is invertible and find its inverse function.

Inverses of Functions, XVI

Example: Verify that the function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $h(x, y) = (x - y, 3y + 1)$ is invertible and find its inverse function.

- To show that h is one-to-one, notice that $h(x_1, y_1) = h(x_2, y_2)$ is the same as $(x_1 - y_1, 3y_1 + 1) = (x_2 - y_2, 3y_2 + 1)$, so by definition of ordered pairs this is equivalent to $x_1 - y_1 = x_2 - y_2$ and $3y_1 + 1 = 3y_2 + 1$.
- The second equation requires $y_1 = y_2$, and then the first equation becomes $x_1 - y_1 = x_2 - y_1$ so $x_1 = x_2$. This means $(x_1, y_1) = (x_2, y_2)$ so h is one-to-one.
- To find the inverse function, we want to solve $h(x, y) = (a, b)$ for (x, y).
- Writing this out yields $(x - y, 3y + 1) = (a, b)$ so $x - y = a$ and $3y + 1 = b$. Solving yields $y = (b - 1)/3$ and then $x = y + a = a + (b - 1)/3$.
- Hence the inverse is $h^{-1}(a, b) = (a + (b - 1)/3, (b - 1)/3)$.

Bijections, I

As we have already seen, functions that are both one-to-one and onto have convenient properties. We now analyze these functions in a bit more detail.

Definition

A function that is both one-to-one and onto is called a bijection.

From our results on inverses, $f : A \to B$ is equivalently a bijection when it has an inverse function $f^{-1} : B \to A$.

Bijections, II

Examples:

- Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 4$ a bijection?

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- Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 4$ a bijection? Yes, because it is both one-to-one and onto.
- Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 3n - 4$ a bijection?

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- Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 4$ a bijection?
  Yes, because it is both one-to-one and onto.
- Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 3n - 4$ a bijection?
  No, although it is one-to-one, it is not onto since there is no n with $f(n) = 0$.
- Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ a bijection?

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- Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 3n - 4$ a bijection? No, although it is one-to-one, it is not onto since there is no n with $f(n) = 0$.
- Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ a bijection? No, it is neither one-to-one nor onto.
- Is the function $j : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash$ given by $j(x) = 1/x$ is a bijection?

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- Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 3n - 4$ a bijection? No, although it is one-to-one, it is not onto since there is no n with $f(n) = 0$.
- Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ a bijection? No, it is neither one-to-one nor onto.
- Is the function $j : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash$ given by $j(x) = 1/x$ is a bijection? No, although it is one-to-one, it is not onto because there is no x for which $j(x) = 0$.
- Is the function $k : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash\{0\}$ given by $k(x) = 1/x$ is a bijection?

Bijections, II

Examples:

- Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 4$ a bijection? Yes, because it is both one-to-one and onto.
- Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 3n - 4$ a bijection? No, although it is one-to-one, it is not onto since there is no n with $f(n) = 0$.
- Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ a bijection? No, it is neither one-to-one nor onto.
- Is the function $j : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash$ given by $j(x) = 1/x$ is a bijection? No, although it is one-to-one, it is not onto because there is no x for which $j(x) = 0$.
- Is the function $k : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash\{0\}$ given by $k(x) = 1/x$ is a bijection? Yes, now the function is one-to-one and onto because 0 has been excluded from the target.

Bijections, III

Example: Determine whether $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) = x^2$ is a bijection.

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- This function f is a bijection. It is one-to-one since $x^2 = y^2$ with x, y positive can only occur for $x = y$, and it is onto since every positive real number has a positive real square root.
- Equivalently, we could observe that f has an inverse function $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f^{-1}(x) = \sqrt{x}$.

Example: Show that there is a bijection between $\mathbb{R}$ and $\mathbb{R}_+$.

Bijections, III

<u>Example</u>: Determine whether $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) = x^2$ is a bijection.

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- Equivalently, we could observe that f has an inverse function $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f^{-1}(x) = \sqrt{x}$.

<u>Example</u>: Show that there is a bijection between $\mathbb{R}$ and $\mathbb{R}_+$.

- We claim that the exponential function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = e^x$ is a bijection.
- To see this, simply observe f has an inverse function $f^{-1} : \mathbb{R}_+ \to \mathbb{R}$ given by the natural logarithm $f^{-1}(x) = \ln x$.

Bijections, IV

If $f : A \to B$ is a bijection, it establishes a one-to-one correspondence between the elements of A and the elements of B.

- Specifically, to each element $a \in A$, f associates a unique element of B – namely $f(a)$ – and to each element $b \in B$, f associates a unique element of A – namely $f^{-1}(b)$.
- We may think of f as being a "relabeling": if we relabel the elements of the set A by applying f to them, then the result is the set B.

Bijections, V

In general, if there exists a bijection $f : A \to B$, we say that A and B are in <u>one-to-one correspondence</u>. This property is, in fact, an equivalence relation.

Proposition (One-to-One Correspondences)

Suppose A, B, and C are sets.

1. *The identity function $i_A : A \to A$ is a bijection from A to A.*
2. *If $f : A \to B$ is a bijection, then its inverse $f^{-1} : B \to A$ is also a bijection.*
3. *If $f : B \to C$ and $g : A \to B$ are bijections, then $f \circ g : A \to C$ is also a bijection.*
4. *The relation on sets defined by $A \sim B$ when there exists a bijection $f : A \to B$ is an equivalence relation.*

Bijections, VI

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Proof:

Bijections, VI

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Proof:

- The identity function is self-evidently one-to-one and onto (alternatively, it is its own inverse).

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2. If $f : A \to B$ is a bijection, then its inverse $f^{-1} : B \to A$ is also a bijection.

Proof:

Bijections, VI

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2. If $f : A \to B$ is a bijection, then its inverse $f^{-1} : B \to A$ is also a bijection.

Proof:

- If $f : A \to B$ is a bijection, then from our results on inverses we know that $f^{-1} : B \to A$ is a function.
- Furthermore, since $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$, we see that f^{-1} is invertible with inverse f, and therefore $f^{-1} : B \to A$ is also a bijection.

Bijections, VII

3. If $f : B \to C$ and $g : A \to B$ are bijections, then $f \circ g : A \to C$ is also a bijection.

Proof:

Bijections, VII

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Proof:

- If $f : B \to C$ and $g : A \to B$ are bijections, then from our results on inverses we know that $f^{-1} : B \to A$ and $g^{-1} : C \to B$ are functions.
- Also, we know that $f \circ g : A \to C$ is invertible with inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1} : C \to A$, so it is also a bijection.

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4. The relation on sets defined by $A \sim B$ when there exists a bijection $f : A \to B$ is an equivalence relation.

Proof:

Bijections, VII

3. If $f : B \to C$ and $g : A \to B$ are bijections, then $f \circ g : A \to C$ is also a bijection.

Proof:

- If $f : B \to C$ and $g : A \to B$ are bijections, then from our results on inverses we know that $f^{-1} : B \to A$ and $g^{-1} : C \to B$ are functions.
- Also, we know that $f \circ g : A \to C$ is invertible with inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1} : C \to A$, so it is also a bijection.

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4. The relation on sets defined by $A \sim B$ when there exists a bijection $f : A \to B$ is an equivalence relation.

Proof:

- This follows immediately from (1)-(3): (1) shows reflexivity, (2) shows symmetry, and (3) shows transitivity.

Summary

We introduced inverse functions and connected them to the properties of being one-to-one and onto.

We established a number of properties of inverse functions.

We introduced the notion of a bijection and gave some basic examples and properties.

Next lecture: Cardinality.

To reiterate the announcement from the start of class, I will be giving Wednesday's lecture in person. Based on how that goes, I will let you know whether in-person lectures will resume permanently.