Math 1365 (Intensive Mathematical Reasoning)

Lecture #23 of 35 $\sim$ November 2, 2023

Function Composition + One-To-One and Onto Functions

- Function Composition
- One-to-One Functions
- Onto Functions

This material represents §3.4.2 + §3.4.3 from the course notes.

Recall, I

Recall some definitions:

Definition

If A and B are sets, a function (or map) from A to B is a relation $f : A \to B$ such that for every $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$, and in such an event we write $f(a) = b$.

The set A is called the domain of f and the set B is called the target (or codomain) of f.

Definition

If $f : A \to B$ is a function, the image of f is the set $\mathrm{im}(f) = \{b \in B : \exists a \in A \text{ with } f(a) = b\}$ of elements $b \in B$ for which there exists at least one $a \in A$ with $f(a) = b$.

Note that the image is always a subset of the target, but need not be equal.

Function Composition, I

Here's the formal definition:

Definition

Let $g : A \to B$ and $f : B \to C$ be functions. Then the *composite function* *$f \circ g : A \to C$ is defined by taking $(f \circ g)(a) = f(g(a))$ for all $a \in A$.*

Function Composition, I

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Let $g : A \to B$ and $f : B \to C$ be functions. Then the <u>composite function</u> $f \circ g : A \to C$ is defined by taking $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

More explicitly, the ordered pairs in $f \circ g$ are those pairs $(a, c) \in A \times C$ for which there exists a $b \in B$ with $(a, b) \in g$ (so that $g(a) = b$) and with $(b, c) \in f$ (so that $f(b) = c$).

- In symbolic language, $f \circ g = \{(a, c) \in A \times C \ : \ \exists b \in B, [(a, b) \in g)] \wedge [(b, c) \in f]\}$.

Remember that function composition is applied right-to-left: in the composition $f \circ g$, the function g is the one that is applied first.

Function Composition, III

When f and g are both described by rules, it is easiest to find compositions using the definition $(f \circ g)(a) = f(g(a))$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions $f(x) = x^2$ and $g(x) = 2x + 1$. Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

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- We have $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2$.
- Likewise $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 + 1$.
- Also, $(f \circ f)(x) = f(f(x)) = f(x^2) = x^4$.
- Finally, $(g \circ g)(x) = g(g(x)) = g(2x + 1) = 4x + 3$.

Function Composition, IV

When f and g are given as sets of ordered pairs, we can use function diagrams to visualize and evaluate compositions: we draw the diagrams for the two functions together, and then follow the arrows from left to right.

- It's very important to make sure that the order of the functions is correct.

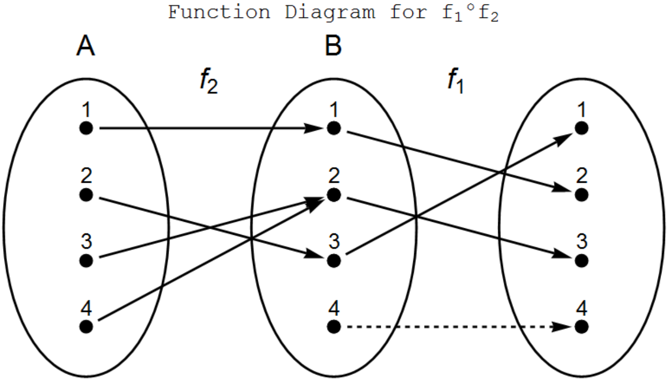
Function Composition, IV

When f and g are given as sets of ordered pairs, we can use function diagrams to visualize and evaluate compositions: we draw the diagrams for the two functions together, and then follow the arrows from left to right.

- It's very important to make sure that the order of the functions is correct.
- Remember that function composition is applied right-to-left: in the composition $f \circ g$, the function g is the one that is applied first.
- This is most easily remembered using the expression $(f \circ g)(x) = f(g(x))$: when evaluating $f(g(x))$, we first calculate $g(x)$, and then we apply f to the result.

Function Composition, V

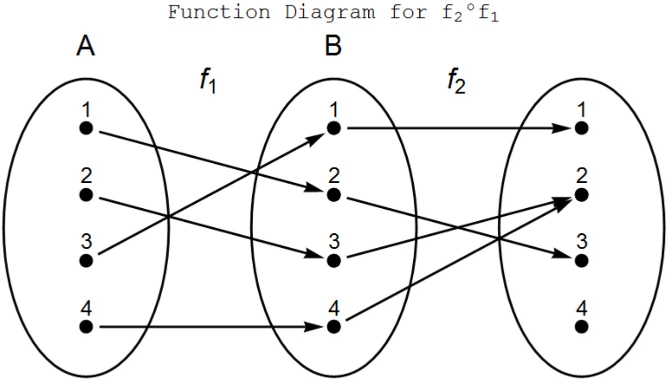
Example: For $f_1 = \{(1,2), (2,3), (3,1), (4,4)\}$ and $f_2 = \{(1,1), (2,3), (3,2), (4,2)\}$ on $\{1,2,3,4\}$, here is a composition diagram for $f_1 \circ f_2$:



We can follow the arrows to see that $(f_1 \circ f_2)(1) = 2$, for instance.

Function Composition, VI

Example: For $f_1 = \{(1,2), (2,3), (3,1), (4,4)\}$ and $f_2 = \{(1,1), (2,3), (3,2), (4,2)\}$ on $\{1,2,3,4\}$, here is a composition diagram for $f_2 \circ f_1$:



We can follow the arrows to see that $(f_2 \circ f_1)(1) = 3$, for instance.

Function Composition, VII

Notice that the result of function composition depends on the order of the functions: in general, it will be the case that $f \circ g$ and $g \circ f$ are completely unrelated functions.

- Indeed, depending on the domains and images of f and g, it is quite possible that one of $f \circ g$ is defined while the other is not.

Function Composition, VIII

For example, suppose $f : \{1, 2\} \to \{a, b\}$ has $f(1) = a$ and $f(2) = b$, and $g : \{a, b\} \to \{3, 4\}$ has $g(a) = 3$ and $g(b) = 4$.

- Then $g \circ f$ exists and is a function from $\{1, 2\}$ to $\{3, 4\}$.
- Specifically, we have $(g \circ f)(1) = g(f(1)) = g(a) = 3$, and $(g \circ f)(2) = g(f(2)) = g(b) = 4$.

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- Then $g \circ f$ exists and is a function from $\{1, 2\}$ to $\{3, 4\}$.
- Specifically, we have $(g \circ f)(1) = g(f(1)) = g(a) = 3$, and $(g \circ f)(2) = g(f(2)) = g(b) = 4$.
- However, $f \circ g$ does not exist.
- The only possible elements in the domain are the elements in the domain of g, but if we try to evaluate $(f \circ g)(a)$, for example, we would have $(f \circ g)(a) = f(g(a)) = f(3)$, and this expression does not make sense because 3 is not in the domain of f.
- Similarly, $(f \circ g)(b) = f(g(b)) = f(4)$ also does not make sense.

Algebra of Function Composition, I

In general, even when both of them are defined, $f \circ g$ and $g \circ f$ are completely unrelated functions.

- For instance, suppose $f(x) = x^2$ and $g(x) = 2x + 1$ as functions from $\mathbb{R}$ to $\mathbb{R}$.
- Then $f(g(x)) = f(2x+1) = (2x+1)^2 = 4x^2 + 4x + 1$ while $g(f(x)) = g(x^2) = 2x^2 + 1$.

This tells us that function composition is NOT commutative: $f \circ g \neq g \circ f$ in general.

Algebra of Function Composition, II

Composition is not commutative, but it does possess some other algebraic properties:

Proposition (Properties of Composition)

Suppose A, B, C, D are sets.

1. *Function composition is associative: If $f : C \to D$, $g : B \to C$, and $h : A \to B$ are any functions then $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as functions from A to D.*
2. *The identity function behaves as a left and right identity: For any $f : A \to B$, $f \circ i_A = f$ and $i_B \circ f = f$.*

Algebra of Function Composition, III

1. Function composition is associative: If $f : C \to D$, $g : B \to C$, and $h : A \to B$ are any functions then $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as functions from A to D.

Proof:

Algebra of Function Composition, III

1. Function composition is associative: If $f : C \to D$, $g : B \to C$, and $h : A \to B$ are any functions then $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as functions from A to D.

Proof:

- The domain of both $(f \circ g) \circ h$ and $f \circ (g \circ h)$ is A, and the target of both $(f \circ g) \circ h$ and $f \circ (g \circ h)$ is D.
- Now let $a \in A$. Then by definition we have $[(f \circ g) \circ h](a) = [(f \circ g)](h(a)) = f(g(h(a)))$, and we also have $[f \circ (g \circ h)](a) = f[(g \circ h)(a)] = f(g(h(a)))$.
- Since these two quantities are equal, we see $[(f \circ g) \circ h](a) = [f \circ (g \circ h)](a)$ for all $a \in A$.
- So, $(f \circ g) \circ h$ and $f \circ (g \circ h)$ have the same domain and target and take the same value at every element of their common domain. So they are the same function.

Algebra of Function Composition, IV

2. The identity function behaves as a left and right identity: For any $f : A \to B$, $f \circ i_A = f$ and $i_B \circ f = f$.

Proof:

Algebra of Function Composition, IV

2. The identity function behaves as a left and right identity: For any $f : A \to B$, $f \circ i_A = f$ and $i_B \circ f = f$.

Proof:

- Observe that the domain of $f \circ i_A$ is A and the target is B, the same as for f.
- Then for any $a \in A$ we have $(f \circ i_A)(a) = f(i_A(a)) = f(a)$, and so we see $f \circ i_A$ and f take the same value at every point of their shared domain. Hence they are equal as functions.
- In the same way, the domain of $i_B \circ f$ is A and the target is B, the same as for f.
- Then for any $a \in A$ we have $(i_B \circ f)(a) = i_B(f(a)) = f(a)$, and so we see $i_B \circ f$ and f take the same value at every point of their shared domain. Hence they are equal as functions.

Motivation for Inverses, I

Next we examine inverses of functions.

- Under the common interpretation of a function f as a "machine" that operates on an input value to produce an output value, the inverse f^{-1} would correspond to a machine that inverts this process, taking an output value of f and giving the corresponding input value.
- In particular, if $f : A \to B$, then we would like to have $f^{-1} : B \to A$, and on the level of ordered pairs, if $(a, b) \in f$, then we would like $(b, a) \in f^{-1}$.

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- In particular, if $f : A \to B$, then we would like to have $f^{-1} : B \to A$, and on the level of ordered pairs, if $(a, b) \in f$, then we would like $(b, a) \in f^{-1}$.
- Indeed, we have already defined an object with this exact property, namely, the inverse relation to f.
- However, if $f : A \to B$ is an arbitrary function, the inverse relation f^{-1} need not be a *function* from B to A.

Motivation for Inverses, II

<u>Example</u>: Suppose $f : \{1, 2, 3\} \to \{5, 6, 7\}$ is given by $f = \{(1, 5), (2, 7), (3, 6)\}$, so that $f(1) = 5$, $f(2) = 7$, and $f(3) = 6$. Find the inverse relation f^{-1}. Is it a function from $\{5, 6, 7\}$ to $\{1, 2, 3\}$?

Motivation for Inverses, II

Example: Suppose $f : \{1, 2, 3\} \to \{5, 6, 7\}$ is given by $f = \{(1, 5), (2, 7), (3, 6)\}$, so that $f(1) = 5$, $f(2) = 7$, and $f(3) = 6$. Find the inverse relation f^{-1}. Is it a function from $\{5, 6, 7\}$ to $\{1, 2, 3\}$?

- Swapping the orders of all the pairs yields the inverse relation $f^{-1} = \{(5, 1), (6, 3), (7, 2)\}$.
- Each of the elements in $\{5, 6, 7\}$ appears in exactly one pair, and all of the second coordinates are in $\{1, 2, 3\}$, so this is a function from $\{5, 6, 7\}$ to $\{1, 2, 3\}$.
- Explicitly, we have $f^{-1}(5) = 1$, $f^{-1}(6) = 3$, and $f^{-1}(7) = 2$.

Motivation for Inverses, III

Example: Suppose $g : \{1, 2, 3, 4\} \to \{5, 6, 7\}$ is given by $g = \{(1, 6), (2, 7), (3, 7), (4, 5)\}$, so that $g(1) = 6$, $g(2) = 7$, $g(3) = 7$, $g(4) = 5$. Find the inverse relation g^{-1}. Is it a function from $\{5, 6, 7\}$ to $\{1, 2, 3, 4\}$?

Motivation for Inverses, III

Example: Suppose $g : \{1, 2, 3, 4\} \to \{5, 6, 7\}$ is given by $g = \{(1, 6), (2, 7), (3, 7), (4, 5)\}$, so that $g(1) = 6$, $g(2) = 7$, $g(3) = 7$, $g(4) = 5$. Find the inverse relation g^{-1}. Is it a function from $\{5, 6, 7\}$ to $\{1, 2, 3, 4\}$?

- Swapping the orders of all the pairs yields the inverse relation $g^{-1} = \{(5, 4), (6, 1), (7, 2), (7, 3)\}$.
- However, g^{-1} is not a function (on any domain) because it contains the ordered pairs $(7, 1)$ and $(7, 3)$, meaning that g^{-1} is not well-defined on the element 7.
- The problem is that g maps both 2 and 3 to 7, so we cannot assign a unique value to $g^{-1}(7)$ since we want it to equal both 2 and 3.

Motivation for Inverses, IV

Example: Suppose $h : \{1, 2, 3\} \to \{5, 6, 7, 8\}$ is given by $h = \{(1, 5), (2, 7), (3, 8)\}$, so that $h(1) = 5$, $h(2) = 7$, and $h(3) = 8$. Find the inverse relation h^{-1}. Is it a function from $\{5, 6, 7, 8\}$ to $\{1, 2, 3\}$?

Motivation for Inverses, IV

Example: Suppose $h : \{1,2,3\} \to \{5,6,7,8\}$ is given by $h = \{(1,5), (2,7), (3,8)\}$, so that $h(1) = 5$, $h(2) = 7$, and $h(3) = 8$. Find the inverse relation h^{-1}. Is it a function from $\{5,6,7,8\}$ to $\{1,2,3\}$?

- Swapping the orders of all the pairs yields the inverse relation $h^{-1} = \{(5,1), (7,2), (8,3)\}$.
- However, h^{-1} is not a function from $\{5,6,7,8\}$ to $\{1,2,3\}$, because there is no pair with first coordinate 6. This is not allowed because then $h^{-1}(6)$ would be undefined.
- In fact, though, h^{-1} is a function from $\{5,7,8\}$ to $\{1,2,3\}$, since all elements of the domain set $\{5,7,8\}$ are the first coordinate in a unique pair.
- Explicitly, we have $h^{-1}(5) = 1$, $h^{-1}(7) = 2$, and $h^{-1}(8) = 3$.
- Notice here that $\mathrm{im}(h) = \{5,7,8\}$, meaning that h^{-1} is a function from $\mathrm{im}(h)$ to the original domain $\{1,2,3\}$ of h.

One-to-One Functions, I

We can clarify this behavior by identifying the precise characteristic of the functions that cause these behaviors:

Definition

The function $f : A \to B$ is one-to-one (or injective) if for any $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

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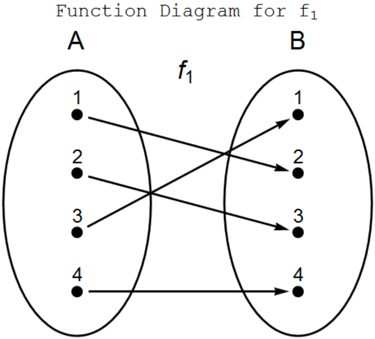
Equivalently, $f : A \to B$ is one-to-one when $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$ – in other words, when f maps unequal elements in its domain to unequal elements in its image.

If we draw a function diagram for f, the definition above says that f is one-to-one whenever we *don't* see two arrows pointing to the same element of the target.

One-to-One Functions, II

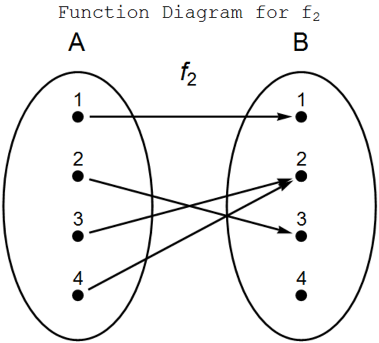
Here's a function diagram for a one-to-one function:



Notice that each of the points on the right has at most one arrow pointing to it.

One-to-One Functions, III

Here's a function diagram for a function that isn't one-to-one:



This function is not one-to-one because it maps 3 and 4 in the domain to the same value 2 in the target.

One-to-One Functions, IV

When a function is defined symbolically via a rule, we can check whether it is one-to-one using the definition.

- Specifically, we want to know whether $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.
- If we think f is one-to-one, to show that we would need to write the equation $f(a_1) = f(a_2)$ and then show that the only solutions are the ones where $a_1 \neq a_2$.
- Alternatively, if we think f isn't one-to-one, we can try looking for counterexamples, which would be a pair (a_1, a_2) with $f(a_1) = f(a_2)$ but $a_1 \neq a_2$.

One-to-One Functions, V

Examples:

1. Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 4$ one-to-one?

One-to-One Functions, V

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2. Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n$ one-to-one?

One-to-One Functions, V

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   Yes: if $g(a_1) = g(a_2)$ then that means $2a_1 = 2a_2$ which upon dividing by 2 yields $a_1 = a_2$.
3. Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^2$ one-to-one?

One-to-One Functions, V

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3. Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^2$ one-to-one? No, because for example $h(2) = h(-2)$ but $2 \neq -2$.
4. Is the function $k : \mathbb{R}_+ \to \mathbb{R}$ given by $k(x) = x^2$ one-to-one?

One-to-One Functions, V

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3. Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^2$ one-to-one? No, because for example $h(2) = h(-2)$ but $2 \neq -2$.
4. Is the function $k : \mathbb{R}_+ \to \mathbb{R}$ given by $k(x) = x^2$ one-to-one? Yes: if $k(x) = k(y)$ so that $x^2 = y^2$, then factoring yields $x = y$ or $x = -y$, but since $x, y > 0$ (since they are in $\mathbb{R}_+$) the second condition cannot occur, and so $x = y$.

One-to-One Functions, VI

As we can see from the last two examples, whether a function is one-to-one depends on the domain, though it doesn't depend on the target.

- Specifically, the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^2$ was not one-to-one, but its restriction $k = h|_{\mathbb{R}+}$ to the positive real numbers is one-to-one.
- So we see that restricting the domain of a function that isn't one-to-one can produce a one-to-one function.

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- So we see that restricting the domain of a function that isn't one-to-one can produce a one-to-one function.

Any restriction of a one-to-one function to a smaller domain will still be one-to-one.

- Specifically, if $f : A \to B$ is one-to-one and $C \subseteq A$, then for $c_1, c_2 \in C$ with $f|_C(c_1) = f|_C(c_2)$ then by definition $f(c_1) = f(c_2)$ and so $c_1 = c_2$.

Onto Functions, I

Another important property of a function is whether its image is the entire target set:

Definition

The function $f : A \to B$ is onto (or surjective) if $\mathrm{im}(f) = B$.

More explicitly, $f : A \to B$ is onto when for any $b \in B$, there exists an $a \in A$ with $f(a) = b$.

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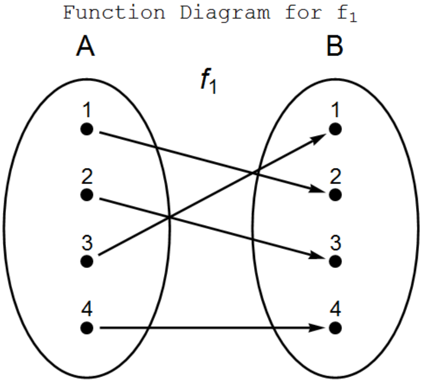
If we draw a function diagram for f, the definition above says that f is onto when every point in the target set has at least one arrow pointing to it.

Onto Functions, II

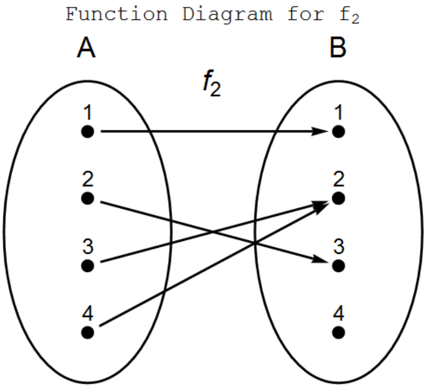
Here's a function diagram for an onto function:



Notice that each of the points on the right has at least one arrow pointing to it.

Onto Functions, III

Here's a function diagram for a function that isn't onto:



This function is not onto because there is no arrow pointing to the value 4 in the target space.

Onto Functions, IV

When a function is defined symbolically via a rule, we can use the definition to check whether it is onto.

- Specifically, we want to know that for every b in the target set B, whether there exists some a in the domain A with $f(a) = b$.
- If we think f is onto, we would need to find (or otherwise show the existence of) such an a for each $b \in B$.
- Alternatively, if we think f isn't onto, we can try looking for counterexamples, which would be an element $b \in B$ such that there exists no $a \in A$ with $f(a) = b$.

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- If we think f is onto, we would need to find (or otherwise show the existence of) such an a for each $b \in B$.
- Alternatively, if we think f isn't onto, we can try looking for counterexamples, which would be an element $b \in B$ such that there exists no $a \in A$ with $f(a) = b$.

The property of being onto requires explicitly knowing the target set for f.

- Every function $f : A \to B$ is surjective onto its image (i.e., when we think of it as a function $f : A \to \mathrm{im}(f)$), but it is not surjective onto any (strictly) larger set.

Onto Functions, V

Examples:

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2. Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n$ onto?

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2. Is the function $g : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n$ onto? No: for example, there is no a with $g(a) = 1$, since the number $1/2$ is not an integer.
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4. Is the function $k : \mathbb{R}_+ \to \mathbb{R}_+$ given by $k(x) = x^2$ onto? Yes: for any positive real number b, there exists a positive real number a, namely, $a = \sqrt{b}$, with $k(a) = b$. (This value of a is found by solving the equation $a^2 = b$ for a.)

Winding Down

Our goal next time is to connect inverse functions with being one-to-one / onto:

Theorem (Inverse Functions)

Suppose $f : A \to B$ is a function. The following are equivalent:

1. *f is one-to-one and onto.*
2. *f^{-1} is a function from B to A.*
3. *There exists a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.*

When any of the conditions (1)-(3) hold, we say f is a bijection from A to B. ("Bijection" = "injection" + "surjection".)

This result gives us a way to know when f has an inverse function, and (when it does have one) how to calculate it.

Summary

We introduced function composition and established some of its properties.

We discussed one-to-one and onto functions and established some of their properties.

We introduced the question of when a function possesses an inverse function.

Next lecture: Inverse functions and bijections.