Math 1365 (Intensive Mathematical Reasoning)

Lecture #22 of 35 $\sim$ November 1, 2023

Functions

- Functions as Relations
- Domain and Image
- Function Composition

This material represents §3.4.1 + §3.4.2 from the course notes.

Functions, I

Our first goal is formalize the definition of a function using the language of relations.

- In elementary school, functions are often explained as being like "machines" that take in an input value and then return a corresponding output value.

Functions, I

Our first goal is formalize the definition of a function using the language of relations.

- In elementary school, functions are often explained as being like "machines" that take in an input value and then return a corresponding output value.
- For instance, the squaring function $f(x) = x^2$ would take the input value 2 and return the value 4, and it would take the input value -1 and return the value 1.
- Indeed, the entire purpose of function notation is to give a clear description of the relation between the input and its corresponding output.

Functions, II

The key idea here is that a function is a description of the *relation* between the set of input values (a set A) and the set of output values (some set B).

- Precisely, for a function f, for each input value $a \in A$ we have a well-defined output value $b = f(a) \in B$.
- We can therefore view f as a relation $f : A \to B$ by saying $(a, b) \in f$ precisely when $f(a) = b$.
- But what makes this relation f a *function*, specifically?

Functions, II

The key idea here is that a function is a description of the *relation* between the set of input values (a set A) and the set of output values (some set B).

- Precisely, for a function f, for each input value $a \in A$ we have a well-defined output value $b = f(a) \in B$.
- We can therefore view f as a relation $f : A \to B$ by saying $(a, b) \in f$ precisely when $f(a) = b$.
- But what makes this relation f a *function*, specifically?
- Just write down the top sentence again, using relation language: for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Functions, III

This is our formal definition of a function:

Definition

If A and B are sets, a function (or map) from A to B is a relation $f : A \to B$ such that for every $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$, and in such an event we write $f(a) = b$.

The set A is called the domain of f and the set B is called the target (or codomain) of f.

We emphasize here that the domain and target are part of the definition of a function. Two functions are equal when their domains are equal, their targets are equal, and their underlying sets of ordered pairs are equal.

Examples of Functions, I

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

1. $f = \{(1, 5),\ (2, 5),\ (3, 7)\}$

Examples of Functions, I

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

1. $f = \{(1, 5),\ (2, 5),\ (3, 7)\}$ is a function from A to B. We have $f(1) = 5$, $f(2) = 5$, and $f(7) = 1$.
2. $g = \{(1, 8),\ (2, 8),\ (3, 8)\}$

Examples of Functions, I

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

1. $f = \{(1, 5), (2, 5), (3, 7)\}$ is a function from A to B. We have $f(1) = 5$, $f(2) = 5$, and $f(7) = 1$.
2. $g = \{(1, 8), (2, 8), (3, 8)\}$ is a function from A to B. We have $g(1) = 8$, $g(2) = 8$, and $g(3) = 8$.
3. $h = \{(1, 5), (2, 8)\}$

Examples of Functions, I

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

1. $f = \{(1, 5), (2, 5), (3, 7)\}$ is a function from A to B. We have $f(1) = 5$, $f(2) = 5$, and $f(7) = 1$.
2. $g = \{(1, 8), (2, 8), (3, 8)\}$ is a function from A to B. We have $g(1) = 8$, $g(2) = 8$, and $g(3) = 8$.
3. $h = \{(1, 5), (2, 8)\}$ is not a function from A to B. The issue is that there is no ordered pair with first coordinate 3 in h, and so the value of $h(3)$ is not defined. (Note that h is a function from $\{1, 2\}$ to B, though.)

Examples of Functions, II

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

4. $k = \{(1, 5), (1, 6), (2, 7), (3, 8)\}$

Examples of Functions, II

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

4. $k = \{(1, 5), (1, 6), (2, 7), (3, 8)\}$ is not a function from A to B. The issue is that there are two pairs with first coordinate 1, so the value of $k(1)$ is not well defined – it tries to define both $k(1) = 5$ and $k(1) = 6$ at the same time, which is not allowed. (In fact, this means k is not a function on any sets.)
5. $l = \{(1, 2), (2, 3), (3, 3)\}$

Examples of Functions, II

Reminder: $f : A \to B$ is a function when for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$.

Consider $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Here are some examples and non-examples of functions $f : A \to B$.

4. $k = \{(1, 5), (1, 6), (2, 7), (3, 8)\}$ is not a function from A to B. The issue is that there are two pairs with first coordinate 1, so the value of $k(1)$ is not well defined – it tries to define both $k(1) = 5$ and $k(1) = 6$ at the same time, which is not allowed. (In fact, this means k is not a function on any sets.)
5. $l = \{(1, 2), (2, 3), (3, 3)\}$ is not a function from A to B. The issue is that the second coordinates do not lie in the set B. (But l is a function from A to A.)

Examples of Functions, III

Here are some more examples of functions:

6. Is the relation $R = \{(1,1), (2,2), (3,3), (4,4)\}$ a function from $\{1,2,3,4\}$ to $\{1,2,3,4\}$?

Examples of Functions, III

Here are some more examples of functions:

6. Is the relation $R = \{(1,1),(2,2),(3,3),(4,4)\}$ a function from $\{1,2,3,4\}$ to $\{1,2,3,4\}$? Yes: each element of the domain is the first coordinate in exactly one pair, and all of the second coordinates are elements of the target.

Examples of Functions, III

Here are some more examples of functions:

6. Is the relation $R = \{(1,1),(2,2),(3,3),(4,4)\}$ a function from $\{1,2,3,4\}$ to $\{1,2,3,4\}$? Yes: each element of the domain is the first coordinate in exactly one pair, and all of the second coordinates are elements of the target.
7. More generally, if A is any set, the identity relation $i_A = \{(a,a) : a \in A\}$ is a function from A to A. For this reason it is called the identity function when we are thinking of it as a function. It has $i_A(a) = a$ for each $a \in A$.

Examples of Functions, III

Here are some more examples of functions:

6. Is the relation $R = \{(1,1),(2,2),(3,3),(4,4)\}$ a function from $\{1,2,3,4\}$ to $\{1,2,3,4\}$? Yes: each element of the domain is the first coordinate in exactly one pair, and all of the second coordinates are elements of the target.
7. More generally, if A is any set, the identity relation $i_A = \{(a,a) : a \in A\}$ is a function from A to A. For this reason it is called the <u>identity function</u> when we are thinking of it as a function. It has $i_A(a) = a$ for each $a \in A$.

Examples of Functions, IV

Identify whether the given relations are or are not functions on the given sets:

8. Is the relation $R = \{(x, x^2) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$?

Examples of Functions, IV

Identify whether the given relations are or are not functions on the given sets:

8. Is the relation $R = \{(x, x^2) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$? Yes, each real number x shows up as the first coordinate in exactly one pair. This function has $f(x) = x^2$ for every real number x – it is the squaring function.
9. Is the relation $S = \{(x^2, x) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$?

Examples of Functions, IV

Identify whether the given relations are or are not functions on the given sets:

8. Is the relation $R = \{(x, x^2) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$? Yes, each real number x shows up as the first coordinate in exactly one pair. This function has $f(x) = x^2$ for every real number x – it is the squaring function.
9. Is the relation $S = \{(x^2, x) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$? No: for instance, it contains the pairs $(4, 2)$ and also $(4, -2)$, but functions are not allowed to have more than one pair with the same first coordinate.

Examples of Functions, IV

Identify whether the given relations are or are not functions on the given sets:

8. Is the relation $R = \{(x, x^2) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$? Yes, each real number x shows up as the first coordinate in exactly one pair. This function has $f(x) = x^2$ for every real number x – it is the squaring function.
9. Is the relation $S = \{(x^2, x) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$? No: for instance, it contains the pairs $(4, 2)$ and also $(4, -2)$, but functions are not allowed to have more than one pair with the same first coordinate.
10. Is the relation $R = \{(2a, a) : a \in \mathbb{Z}\}$ a function from $\mathbb{Z}$ to $\mathbb{Z}$?

Examples of Functions, IV

Identify whether the given relations are or are not functions on the given sets:

8. Is the relation $R = \{(x, x^2) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$?
   Yes, each real number x shows up as the first coordinate in exactly one pair. This function has $f(x) = x^2$ for every real number x – it is the squaring function.
9. Is the relation $S = \{(x^2, x) : x \in \mathbb{R}\}$ a function from $\mathbb{R}$ to $\mathbb{R}$?
   No: for instance, it contains the pairs $(4, 2)$ and also $(4, -2)$, but functions are not allowed to have more than one pair with the same first coordinate.
10. Is the relation $R = \{(2a, a) : a \in \mathbb{Z}\}$ a function from $\mathbb{Z}$ to $\mathbb{Z}$?
    No: for instance, there is no pair with first coordinate 1 (a valid element of the domain $\mathbb{Z}$), so this relation is not a function from $\mathbb{Z}$ to $\mathbb{Z}$.

Examples of Functions, V

Let's now consider the empty relation $E = \{\,\} = \emptyset$.

11. Is the empty relation a function from $\{1, 2\}$ to $\{1, 2, 3\}$?

Examples of Functions, V

Let's now consider the empty relation $E = \{\,\} = \emptyset$.

11. Is the empty relation a function from $\{1, 2\}$ to $\{1, 2, 3\}$?
    No, because none of the elements in the domain shows up as the first coordinate of any pairs in E.
12. Is the empty relation a function from $\{1, 2\}$ to $\{\,\} = \emptyset$?

Examples of Functions, V

Let's now consider the empty relation $E = \{\,\} = \emptyset$.

11. Is the empty relation a function from $\{1, 2\}$ to $\{1, 2, 3\}$?
    No, because none of the elements in the domain shows up as the first coordinate of any pairs in E.
12. Is the empty relation a function from $\{1, 2\}$ to $\{\,\} = \emptyset$?
    No, for the same reason as before: none of the elements in the domain shows up as the first coordinate of any pairs in E.
13. Is the empty relation a function from $\{\,\} = \emptyset$ to $\{\,\} = \emptyset$?

Examples of Functions, V

Let's now consider the empty relation $E = \{\,\} = \emptyset$.

11. Is the empty relation a function from $\{1, 2\}$ to $\{1, 2, 3\}$?
    No, because none of the elements in the domain shows up as the first coordinate of any pairs in E.
12. Is the empty relation a function from $\{1, 2\}$ to $\{\,\} = \emptyset$?
    No, for the same reason as before: none of the elements in the domain shows up as the first coordinate of any pairs in E.
13. Is the empty relation a function from $\{\,\} = \emptyset$ to $\{\,\} = \emptyset$?
    Yes: every element of the domain $\{\,\} = \emptyset$ shows up as the first coordinate in exactly one pair in E, and all second coordinates of pairs in E are elements of the target set $\{\,\} = \emptyset$.
14. Is the empty relation a function from $\{\,\} = \emptyset$ to $\{1, 2, 3\}$?

Examples of Functions, V

Let's now consider the empty relation $E = \{\,\} = \emptyset$.

11. Is the empty relation a function from $\{1, 2\}$ to $\{1, 2, 3\}$?
    No, because none of the elements in the domain shows up as the first coordinate of any pairs in E.
12. Is the empty relation a function from $\{1, 2\}$ to $\{\,\} = \emptyset$?
    No, for the same reason as before: none of the elements in the domain shows up as the first coordinate of any pairs in E.
13. Is the empty relation a function from $\{\,\} = \emptyset$ to $\{\,\} = \emptyset$?
    Yes: every element of the domain $\{\,\} = \emptyset$ shows up as the first coordinate in exactly one pair in E, and all second coordinates of pairs in E are elements of the target set $\{\,\} = \emptyset$.
14. Is the empty relation a function from $\{\,\} = \emptyset$ to $\{1, 2, 3\}$?
    Yes: every element of the domain $\{\,\} = \emptyset$ shows up as the first coordinate in exactly one pair in E, and all second coordinates of pairs in E are elements of the target set $\{1, 2, 3\}$.

Functions Defined By Rules, I

Many functions can be defined by a general rule or description.

- For instance, for the function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ given by $f = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, we could abbreviate the definition by saying $f(n) = 5 - n$ for all $n \in \{1, 2, 3, 4\}$.
- Most of the time, we would just write this function as $f(n) = 5 - n$, with the implicit assumption that this rule is valid for all n in the domain of f, which here is $\{1, 2, 3, 4\}$.

Warning: when we abbreviate definitions in this manner, we MUST still specify what the domain of the function is!

Functions Defined By Rules, I

Many functions can be defined by a general rule or description.

- For instance, for the function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ given by $f = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, we could abbreviate the definition by saying $f(n) = 5 - n$ for all $n \in \{1, 2, 3, 4\}$.
- Most of the time, we would just write this function as $f(n) = 5 - n$, with the implicit assumption that this rule is valid for all n in the domain of f, which here is $\{1, 2, 3, 4\}$.

Warning: when we abbreviate definitions in this manner, we MUST still specify what the domain of the function is!

- For instance, if $g : \{4, 5\} \to \{0, 1, 2\}$ is the function $g = \{(4, 1), (5, 0)\}$, then we could also abbreviate the definition of g as $g(n) = 5 - n$.
- This definition looks the same as the one for f above, but f and g are not equal! (Just look at the ordered pairs.)

Functions Defined By Rules, II

When defining a function $f : A \to B$ by a rule or description, the definition must be unambiguous and yield a well-defined value $b = f(a)$ for each $a \in A$.

- In some situations, the ambiguities in a definition might not be obvious.
- For instance, suppose we attempt to define a "numerator" function $f : \mathbb{Q} \to \mathbb{Z}$ by saying $f(a/b) = a$ for any $a/b \in \mathbb{Q}$.

Functions Defined By Rules, II

When defining a function $f : A \to B$ by a rule or description, the definition must be unambiguous and yield a well-defined value $b = f(a)$ for each $a \in A$.

- In some situations, the ambiguities in a definition might not be obvious.
- For instance, suppose we attempt to define a "numerator" function $f : \mathbb{Q} \to \mathbb{Z}$ by saying $f(a/b) = a$ for any $a/b \in \mathbb{Q}$.
- Although this may appear to be valid (the formula gives a clear, explicit value for each input a/b), it does not actually yield a well-defined function.
- Why not? Well, per the rule given, we would have $f(1/2) = 1$ while $f(2/4) = 2$, but $1/2 = 2/4$ as rational numbers. On the level of ordered pairs, f would contain both $(1/2, 1)$ and $(2/4, 2) = (1/2, 2)$, which is not allowed.

Functions Defined By Rules, III

How could we fix the previous attempted definition of a "numerator" function $f : \mathbb{Q} \to \mathbb{Z}$ with $f(a/b) = a$ for any $a/b \in \mathbb{Q}$?

- We would need to address the fact that every rational number has many different equivalent expressions.

Functions Defined By Rules, III

How could we fix the previous attempted definition of a "numerator" function $f : \mathbb{Q} \to \mathbb{Z}$ with $f(a/b) = a$ for any $a/b \in \mathbb{Q}$?

- We would need to address the fact that every rational number has many different equivalent expressions.
- We could, for instance, clarify the definition by saying that $f(a/b) = a$ only when $a/b \in \mathbb{Q}$ is in lowest terms (meaning that a and b are relatively prime and $b > 0$).
- Since each rational number can be written uniquely in the form a/b in lowest terms where $b > 0$ (you may find it exciting to prove this fact yourself) the definition is now unambiguous.
- For instance, $f(3/6) = 1$ because $3/6 = 1/2$ in lowest terms.

Representing Functions Visually, I

We have various convenient ways to represent functions visually.

- One visual representation you are likely familiar with is the graph of a function f, which is obtained by drawing all ordered pairs $(x, y) \in f$.

Representing Functions Visually, I

We have various convenient ways to represent functions visually.

- One visual representation you are likely familiar with is the graph of a function f, which is obtained by drawing all ordered pairs $(x, y) \in f$.
- In fact, we don't even need f to be a function to draw its graph: we can just as well draw the graph of a relation R (just plot all pairs $(x, y) \in R$).

But for the kinds of things we will be doing with functions, these kinds of graphs are not very useful. (They're far more useful if you want to do calculus, or something like that.)

Representing Functions Visually, II

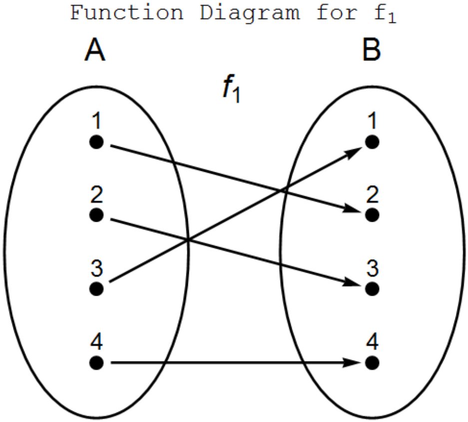
We're interested in functions $f : A \to B$ for arbitrary sets A and B.

- For functions $f : A \to B$ defined on finite sets, or sets that do not consist of real numbers, the graph is typically either not useful, or not possible to draw sensibly.
- For this reason, we instead use "relation diagrams", in which we represent the sets A and B as collections of points and draw an arrow from $a \in A$ to $b \in B$ whenever $(a, b) \in f$.

These relation diagrams (which for functions we usually call "function diagrams") will be useful later when we start describing properties of functions.

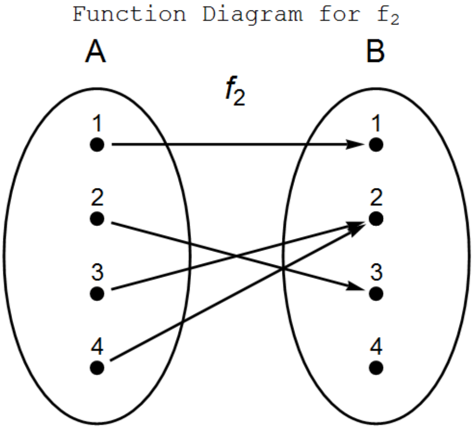
Representing Functions Visually, III

Here's a function diagram for $f_1 = \{(1,2),(2,3),(3,1),(4,4)\}$ from $\{1,2,3,4\}$ to $\{1,2,3,4\}$:



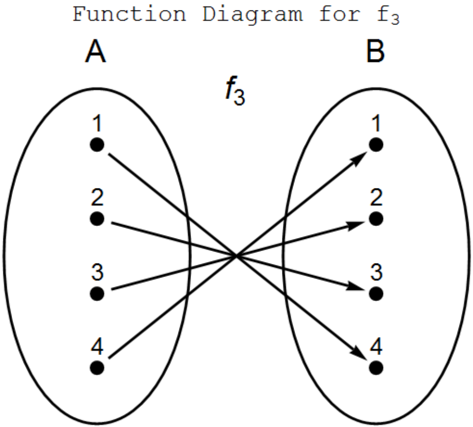
Representing Functions Visually, IV

Here's a function diagram for $f_2 = \{(1,1), (2,3), (3,2), (4,2)\}$ from $\{1,2,3,4\}$ to $\{1,2,3,4\}$:



Representing Functions Visually, V

Here's a function diagram for $f_3 = \{(1,4), (2,3), (3,2), (4,1)\}$ from $\{1,2,3,4\}$ to $\{1,2,3,4\}$:



Image, I

An important property of a function is its set of "output values":

Definition

If $f : A \to B$ is a function, the image of f is the set $\mathrm{im}(f) = \{b \in B : \exists a \in A \text{ with } f(a) = b\}$ of elements $b \in B$ for which there exists at least one $a \in A$ with $f(a) = b$.

Terminology Note: Some people use the word "range" as a synonym for "codomain" / "'target", while others use it as synonym for "image". We will avoid using the word "range" for this reason.

Image, II

Examples: For each function from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, 4\}$, find its image (i.e., the set of output values):

1. $f = \{(1, 1), (2, 1), (3, 1), (4, 4)\}$.
2. $g = \{(1, 1), (2, 3), (3, 2), (4, 2)\}$.
3. $h(n) = 5 - n$.
4. $k(n) = \begin{cases} 1 \text{ when } n \text{ is odd} \\ 2 \text{ when } n \text{ is even} \end{cases}$.

Image, II

<u>Examples</u>: For each function from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, 4\}$, find its image (i.e., the set of output values):

1. $f = \{(1, 1), (2, 1), (3, 1), (4, 4)\}$.
   - The image is $\{1, 4\}$.
2. $g = \{(1, 1), (2, 3), (3, 2), (4, 2)\}$.
   - The image is $\{1, 2, 3\}$.
3. $h(n) = 5 - n$.
   - The image is $\{1, 2, 3, 4\}$.
4. $k(n) = \begin{cases} 1 \text{ when } n \text{ is odd} \\ 2 \text{ when } n \text{ is even} \end{cases}$.
   - The image is $\{1, 2\}$.

Image, III

We emphasize that the image of a function $f : A \to B$ is always a subset of the target set B, but need not be equal in general.

- For example, the image of $f = \{(1,1), (2,1), (3,1), (4,4)\}$ from the last slide was the set $\{1,4\}$ even though the target set was $\{1,2,3,4\}$.

Image, III

We emphasize that the image of a function $f : A \to B$ is always a subset of the target set B, but need not be equal in general.

- For example, the image of $f = \{(1,1), (2,1), (3,1), (4,4)\}$ from the last slide was the set $\{1,4\}$ even though the target set was $\{1,2,3,4\}$.
- As another example, the image of the function $h : \mathbb{Z} \to \mathbb{Z}$ with $h(n) = 2n$ is the set of even integers.
- As a third example, the image of the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$ is the set $\mathbb{R}_{\geq 0}$ of nonnegative real numbers.

Restrictions of Functions, I

Since we view functions as relations, all of the operations we can perform with relations can also be performed on functions.

One important operation is that of restricting a function to a smaller domain; since this operation on functions is particularly useful, we (re-)record the definition explicitly:

Definition

If C is a subset of A and $f : A \to B$ is a function, the restriction *of f to the domain C, denoted $f|_C$, is the function $f|_C : C \to B$ given by $f|_C = f \cap (C \times B)$.*

Restrictions of Functions, I

Since we view functions as relations, all of the operations we can perform with relations can also be performed on functions.

One important operation is that of restricting a function to a smaller domain; since this operation on functions is particularly useful, we (re-)record the definition explicitly:

Definition

If C is a subset of A and $f : A \to B$ is a function, the restriction of f to the domain C, denoted $f|_C$, is the function $f|_C : C \to B$ given by $f|_C = f \cap (C \times B)$.

The ordered pairs in $f|_C$ are those of the form (c, b) where $c \in C$ and $(c, b) \in f$.

- We can think of $f|_C$ as the function obtained by "throwing away" the information about the values on f on the elements of A not in C.

Restrictions of Functions, II

Example:

- For $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with $f = \{(1, 2),\ (2, 3),\ (3, 1),\ (4, 4)\}$, the restriction of f to the domain $\{1, 3\}$ is the function $g : \{1, 3\} \to \{1, 2, 3, 4\}$ with $g = \{(1, 2),\ (3, 1)\}$.

Restrictions of Functions, II

Example:

- For $f : \{1,2,3,4\} \to \{1,2,3,4\}$ with $f = \{(1,2),\ (2,3),\ (3,1),\ (4,4)\}$, the restriction of f to the domain $\{1,3\}$ is the function $g : \{1,3\} \to \{1,2,3,4\}$ with $g = \{(1,2),\ (3,1)\}$.

In the particular situation where f is defined using a rule, we simply use the same rule for $f|_C$ on the smaller domain C.

- For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, we may restrict f to the positive real numbers to obtain a new function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(x) = x^2$.

Restrictions of Functions, III

In some situations we can restrict or enlarge the target set.

- Indeed, if $f : A \to B$ is a function with image $\mathrm{im}(f)$, then we also have a function $g : A \to \mathrm{im}(f)$ given by the same collection of ordered pairs, whose target set is now $\mathrm{im}(f)$.
- More generally, if C is any set with $\mathrm{im}(f) \subseteq C$, we may also view the same collection of ordered pairs as yielding a function $h : A \to C$.

Restrictions of Functions, III

In some situations we can restrict or enlarge the target set.

- Indeed, if $f : A \to B$ is a function with image $\mathrm{im}(f)$, then we also have a function $g : A \to \mathrm{im}(f)$ given by the same collection of ordered pairs, whose target set is now $\mathrm{im}(f)$.
- More generally, if C is any set with $\mathrm{im}(f) \subseteq C$, we may also view the same collection of ordered pairs as yielding a function $h : A \to C$.

It is a matter of taste whether to consider this function h as being "the same as" f, since its underlying collection of ordered pairs, domain, and image are the same as f's.

- In practice, it is common to view this function as being equivalent to f, since it carries the same information.
- We have adopted the convention that the domain and target are parts of the definition of a function. So we would not consider h to be equal to f, since its target set is different.

Function Composition, I

We now discuss ways of constructing new functions from other functions, of which the most fundamental is function composition.

- Informally, if f and g are functions, the notation $f(g(x))$ is used to symbolize the result of applying f to the value $g(x)$.
- This operation is well-defined provided that the image of g is a subset of the domain of f.
- We use the notation $f \circ g$ to refer to the composite function itself, so that $(f \circ g)(x) = f(g(x))$.

Function Composition, II

Here's the formal definition:

Definition

Let $g : A \to B$ and $f : B \to C$ be functions. Then the <u>*composite function*</u> *$f \circ g : A \to C$ is defined by taking $(f \circ g)(a) = f(g(a))$ for all $a \in A$.*

Function Composition, II

Here's the formal definition:

Definition

Let $g : A \to B$ and $f : B \to C$ be functions. Then the <u>composite function</u> $f \circ g : A \to C$ is defined by taking $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

More explicitly, the ordered pairs in $f \circ g$ are those pairs $(a, c) \in A \times C$ for which there exists a $b \in B$ with $(a, b) \in g$ (so that $g(a) = b$) and with $(b, c) \in f$ (so that $f(b) = c$).

- In symbolic language, $f \circ g = \{(a, c) \in A \times C \, : \, \exists b \in B, [(a, b) \in g)] \wedge [(b, c) \in f]\}$.

Function Composition, III

When f and g are both described by rules, it is easiest to find compositions using the definition $(f \circ g)(a) = f(g(a))$.

<u>Example</u>: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions $f(x) = x^2$ and $g(x) = 2x + 1$. Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

Function Composition, III

When f and g are both described by rules, it is easiest to find compositions using the definition $(f \circ g)(a) = f(g(a))$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions $f(x) = x^2$ and $g(x) = 2x + 1$. Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

- We have $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2$.
- Likewise $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 + 1$.
- Also, $(f \circ f)(x) = f(f(x)) = f(x^2) = x^4$.
- Finally, $(g \circ g)(x) = g(g(x)) = g(2x + 1) = 4x + 3$.

Function Composition, IV

When f and g are given as sets of ordered pairs, we can use function diagrams to visualize and evaluate compositions: we draw the diagrams for the two functions together, and then follow the arrows from left to right.

- It's very important to make sure that the order of the functions is correct.

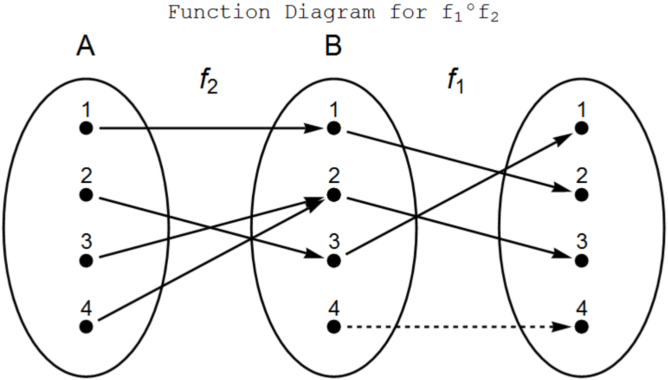
Function Composition, IV

When f and g are given as sets of ordered pairs, we can use function diagrams to visualize and evaluate compositions: we draw the diagrams for the two functions together, and then follow the arrows from left to right.

- It's very important to make sure that the order of the functions is correct.
- Remember that function composition is applied right-to-left: in the composition $f \circ g$, the function g is the one that is applied first.
- This is most easily remembered using the expression $(f \circ g)(x) = f(g(x))$: when evaluating $f(g(x))$, we first calculate $g(x)$, and then we apply f to the result.

Function Composition, V

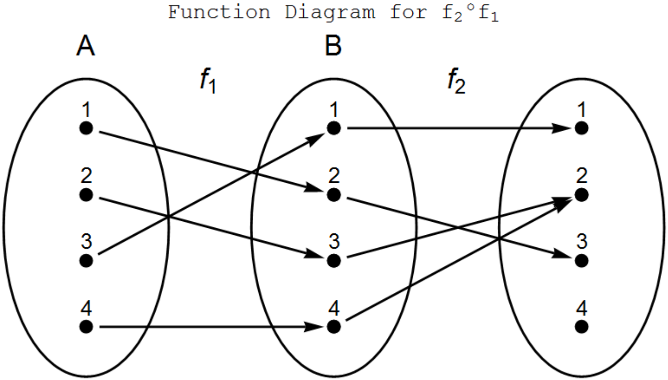
Example: For $f_1 = \{(1,2), (2,3), (3,1), (4,4)\}$ and $f_2 = \{(1,1), (2,3), (3,2), (4,2)\}$ on $\{1,2,3,4\}$, here is a composition diagram for $f_1 \circ f_2$:



We can follow the arrows to see that $(f_1 \circ f_2)(1) = 2$, for instance.

Function Composition, VI

<u>Example</u>: For $f_1 = \{(1,2), (2,3), (3,1), (4,4)\}$ and $f_2 = \{(1,1), (2,3), (3,2), (4,2)\}$ on $\{1,2,3,4\}$, here is a composition diagram for $f_2 \circ f_1$:



We can follow the arrows to see that $(f_2 \circ f_1)(1) = 3$, for instance.

Function Composition, VII

Notice that the result of function composition depends on the order of the functions: in general, it will be the case that $f \circ g$ and $g \circ f$ are completely unrelated functions.

- Indeed, depending on the domains and images of f and g, it is quite possible that one of $f \circ g$ is defined while the other is not.

Function Composition, VIII

For example, suppose $f : \{1, 2\} \to \{a, b\}$ has $f(1) = a$ and $f(2) = b$, and $g : \{a, b\} \to \{3, 4\}$ has $g(a) = 3$ and $g(b) = 4$.

- Then $g \circ f$ exists and is a function from $\{1, 2\}$ to $\{3, 4\}$.
- Specifically, we have $(g \circ f)(1) = g(f(1)) = g(a) = 3$, and $(g \circ f)(2) = g(f(2)) = g(b) = 4$.

Function Composition, VIII

For example, suppose $f : \{1, 2\} \to \{a, b\}$ has $f(1) = a$ and $f(2) = b$, and $g : \{a, b\} \to \{3, 4\}$ has $g(a) = 3$ and $g(b) = 4$.

- Then $g \circ f$ exists and is a function from $\{1, 2\}$ to $\{3, 4\}$.
- Specifically, we have $(g \circ f)(1) = g(f(1)) = g(a) = 3$, and $(g \circ f)(2) = g(f(2)) = g(b) = 4$.
- However, $f \circ g$ does not exist.
- The only possible elements in the domain are the elements in the domain of g, but if we try to evaluate $(f \circ g)(a)$, for example, we would have $(f \circ g)(a) = f(g(a)) = f(3)$, and this expression does not make sense because 3 is not in the domain of f.
- Similarly, $(f \circ g)(b) = f(g(b)) = f(4)$ also does not make sense.

Summary

We discussed functions as relations and gave various examples.

We discussed the image of a function and how to restrict the domain of a function.

We introduced function composition.

Next lecture: Properties of function composition, inverses of functions.