Math 1365 (Intensive Mathematical Reasoning)

Lecture #21 of 35 $\sim$ October 30, 2023

Smallest, Largest, Minimal, and Maximal Elements

- Smallest and Largest Elements
- Minimal and Maximal Elements
- Properties of Smallest, Largest, Minimal, Maximal Elements

This material represents §3.3.2 from the course notes.

Recall

Recall the definition of a partial ordering from last class:

Definition

Suppose R is a relation on a set A.

R is reflexive when $a\ R\ a$ for all $a \in A$.

R is antisymmetric when $a\ R\ b$ and $b\ R\ a$ imply that $a = b$.

R is transitive when $a\ R\ b$ and $b\ R\ c$ imply that $a\ R\ c$.

When is R is reflexive, antisymmetric, and transitive, we say R is a partial ordering of A (or partial order on A).

If it is also true that for any $a, b \in A$ at least one of $a\ R\ b$ and $b\ R\ a$ is true, we say R is a total ordering (or linear ordering).

Recall, II

Some common examples of partial and total orderings are as follows:

- The relation $\leq$ on real numbers is a total ordering.

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- The relation $\leq$ on real numbers is a total ordering.
- The relation $\geq$ on real numbers is also a total ordering.
- The divisibility relation $|$ on positive integers is a partial ordering, but not a total ordering.

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- The relation $\geq$ on real numbers is also a total ordering.
- The divisibility relation $|$ on positive integers is a partial ordering, but not a total ordering.
- The subset relation on sets (inside a universal set U) is a partial ordering, but not a total ordering.

Recall, II

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- The relation $\leq$ on real numbers is a total ordering.
- The relation $\geq$ on real numbers is also a total ordering.
- The divisibility relation $|$ on positive integers is a partial ordering, but not a total ordering.
- The subset relation on sets (inside a universal set U) is a partial ordering, but not a total ordering.
- The alphabetical dictionary ordering on the set of English words is a total ordering.

Discussion, I

A partial ordering formalizes the idea of comparing the "size" of two elements in a set.

- When R is a partial ordering on A, we can think of the statement $a\ R\ b$ as saying "a is at most as big as b", or the reverse "b is at least as big as a".
- This is precisely the meaning of the statement $a \leq b$, and in fact, partial orderings in general are often written using the symbol $\leq$ or something that looks like it, even in general contexts (much like how equivalence relations are often written as $\sim$).

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- This is precisely the meaning of the statement $a \leq b$, and in fact, partial orderings in general are often written using the symbol $\leq$ or something that looks like it, even in general contexts (much like how equivalence relations are often written as $\sim$).
- For instance, that is why we use the symbol $\subseteq$ for the subset relation: to remind you that $\subseteq$ has similar properties to $\leq$.

Discussion, II

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- In other words, among the two elements a and b, one of them is always bigger than the other with respect to the ordering R (or possibly, they are equal).

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In a total ordering, the additional condition that for any a, b in A at least one of $a \; R \; b$ and $b \; R \; a$ is true says that any two elements are always comparable.

- In other words, among the two elements a and b, one of them is always bigger than the other with respect to the ordering R (or possibly, they are equal).
- Precisely: when $a \; R \; b$ is true then b is bigger, while when $b \; R \; a$ is true then a is bigger. (When both $a \; R \; b$ and $b \; R \; a$ are true, then by antisymmetry, $a = b$.)

In many contexts when we are working with partial and total orderings, important properties are often attached to extremal elements: elements that are the smallest or the largest with respect to the ordering.

Smallest and Largest, I

We formalize these notions as follows:

Definition

Suppose R is a partial ordering on a set A.

We say that an element $x \in A$ is a <u>smallest element</u> (or <u>least element</u>) of A with respect to R when $x \, R \, a$ for all $a \in A$.

We say $x \in R$ is a <u>largest element</u> (or <u>greatest element</u>) of A with respect to R if $a \, R \, x$ for all $a \in A$.

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Suppose R is a partial ordering on a set A.

We say that an element $x \in A$ is a smallest element (or least element) of A with respect to R when $x\,R\,a$ for all $a \in A$.

We say $x \in R$ is a largest element (or greatest element) of A with respect to R if $a\,R\,x$ for all $a \in A$.

Smallest and largest elements need not exist, as you will see in the examples on the next slides. But when they do exist, they are unique (as we will show very soon): thus we may refer to *the* smallest element rather than merely *a* smallest element.

Smallest and Largest, II

Reminders:
x is smallest when $x \, R \, a$ for all $a \in A$.
x is largest when $a \, R \, x$ for all $a \in A$.

Examples: For each partial ordering on each set, identify the smallest and largest elements, if they exist.

1. The set $\{1, 2, 3, 6, 10\}$ with ordering $\leq$.

2. The set $\{1, 2, 3, 5, 6\}$ with the divisibility ordering $|$.

3. The set $\{3, 4, 5, 6, 7, 8\}$ with the divisibility ordering $|$.

4. The subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.

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   - Smallest: 1. Largest: 10.
2. The set $\{1, 2, 3, 5, 6\}$ with the divisibility ordering $|$.
   - Smallest: 1. Largest: does not exist.
3. The set $\{3, 4, 5, 6, 7, 8\}$ with the divisibility ordering $|$.
   - Smallest: does not exist. Largest: does not exist.
4. The subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.
   - Smallest: $\{\} = \emptyset$. Largest: $\{1, 2, 3, 4, 5\}$.

Smallest and Largest, III

Reminders:
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x is largest when $a \; R \; x$ for all $a \in A$.

Examples: For each partial ordering on each set, identify the smallest and largest elements, if they exist.

5. Nonempty subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.
6. The set of positive integers with ordering $\leq$.
7. The set of positive real numbers with ordering $\leq$.
8. The set $\mathbb{Z}$ of integers with ordering $\leq$.

Smallest and Largest, III

Reminders:
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Examples: For each partial ordering on each set, identify the smallest and largest elements, if they exist.

5. Nonempty subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.
   - Smallest: does not exist. Largest: $\{1, 2, 3, 4, 5\}$.
6. The set of positive integers with ordering $\leq$.
   - Smallest: 1. Largest: does not exist.
7. The set of positive real numbers with ordering $\leq$.
   - Smallest: does not exist. Largest: does not exist.
8. The set $\mathbb{Z}$ of integers with ordering $\leq$.
   - Smallest: does not exist. Largest: does not exist.

Smallest and Largest, IV

Let's now prove that there is always at most one smallest and at most one largest element:

Proposition (Smallest and Largest)

Let A be a set.

1. *If R is a partial ordering on A, then there is at most one smallest element of A and at most one largest element of A.*
2. *If R is a total ordering on A, and A is finite and nonempty, then there is exactly one smallest and exactly one largest element.*

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You get to prove (2) yourself on Homework 8. (A good approach is to use induction on the number of elements in the set.)

Smallest and Largest, V

1. If R is a partial ordering on A, then there is at most one smallest element of A and at most one largest element of A.

Proof:

Smallest and Largest, V

1. If R is a partial ordering on A, then there is at most one smallest element of A and at most one largest element of A.

Proof:

- First suppose x and y are both smallest elements of A.
- Then $x\,R\,y$ since x is smallest.
- Also, $y\,R\,x$ since y is smallest.
- But then by antisymmetry, $x = y$.
- A very similar argument works for largest elements.

Minimal and Maximal, I

Closely related to smallest and largest elements are minimal and maximal elements:

Definition

Suppose R is a partial ordering on a set A.

We say that an element $x \in A$ is a <u>minimal element</u> of A with respect to R (or just <u>minimal</u>) when $y \, R \, x$ implies $y = x$.

We say $x \in A$ is a <u>maximal element</u> of A with respect to R (or just <u>maximal</u>) when $x \, R \, y$ implies $y = x$.

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Closely related to smallest and largest elements are minimal and maximal elements:

Definition

Suppose R is a partial ordering on a set A.

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We say $x \in A$ is a maximal element of A with respect to R (or just maximal) when $x \, R \, y$ implies $y = x$.

The intuition for these definitions is as follows:

- An element x is minimal when there is nothing strictly below it: i.e., there is no y with $y \, R \, x$ and $y \neq x$.
- An element x is maximal when there is nothing strictly above it: i.e., there is no y with $x \, R \, y$ and $y \neq x$.

Minimal and Maximal, II

Reminders:
x is minimal when $y \, R \, x$ implies $y = x$.
x is maximal when $x \, R \, y$ implies $y = x$.

Examples: For each partial ordering on each set, identify all minimal and maximal elements, if they exist.

1. The set $\{1, 2, 3, 6, 10\}$ with ordering $\leq$.

2. The set $\{1, 2, 3, 5, 6\}$ with the divisibility ordering $|$.

3. The set $\{3, 4, 5, 6, 7, 8\}$ with the divisibility ordering $|$.

4. The subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.

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Reminders:
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<u>Examples</u>: For each partial ordering on each set, identify all minimal and maximal elements, if they exist.

1. The set $\{1, 2, 3, 6, 10\}$ with ordering $\leq$.
   - Minimal: 1. Maximal: 10.
2. The set $\{1, 2, 3, 5, 6\}$ with the divisibility ordering $|$.
   - Minimal: 1. Maximal: 5, 6.
3. The set $\{3, 4, 5, 6, 7, 8\}$ with the divisibility ordering $|$.
   - Minimal: 3, 4, 5, 7. Maximal: 5, 6, 7, 8.
4. The subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.
   - Minimal: $\{\} = \emptyset$. Maximal: $\{1, 2, 3, 4, 5\}$.

Minimal and Maximal, III

Reminders:
x is minimal when $y \, R \, x$ implies $y = x$.
x is maximal when $x \, R \, y$ implies $y = x$.

Examples: For each partial ordering on each set, identify all minimal and maximal elements, if they exist.

5. Nonempty subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.
6. The set of positive integers with ordering $\leq$.
7. The set of positive real numbers with ordering $\leq$.
8. The set $\mathbb{Z}$ of integers with ordering $\leq$.

Minimal and Maximal, III

Reminders:
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Examples: For each partial ordering on each set, identify all minimal and maximal elements, if they exist.

5. Nonempty subsets of $\{1, 2, 3, 4, 5\}$ with the subset ordering $\subseteq$.
   - Minimal: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$. Maximal: $\{1, 2, 3, 4, 5\}$.
6. The set of positive integers with ordering $\leq$.
   - Minimal: 1. Maximal: does not exist.
7. The set of positive real numbers with ordering $\leq$.
   - Minimal: does not exist. Maximal: does not exist.
8. The set $\mathbb{Z}$ of integers with ordering $\leq$.
   - Minimal: does not exist. Maximal: does not exist.

Minimal and Maximal, IV

The notions of minimal element ("nothing is smaller than x") and smallest element ("everything is greater than x") capture very similar ideas, although as we can see from the examples, they are not the same.

- For instance, with the divisibility relation on $\{1, 2, 3, 5, 6\}$, we saw that 1 was the smallest element and also the only minimal element.

Minimal and Maximal, IV

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- For instance, with the divisibility relation on $\{1, 2, 3, 5, 6\}$, we saw that 1 was the smallest element and also the only minimal element.
- But for the divisibility relation on $\{3, 4, 5, 6, 7, 8\}$, there was no smallest element, yet 3, 4, 5, and 7 were all minimal.

Likewise, being maximal is very similar to being greatest, though again, they're not always the same.

Minimal and Maximal, V

In fact, if there is a smallest element then it will be the unique minimal element, and if there is a largest element then it will be the unique maximal element:

Proposition (Minimal and Maximal Properties)

3. *Let R be a partial ordering on a set A. If $x \in A$ is smallest then x is the unique minimal element of A, and if $x \in A$ is largest then x is the unique maximal element of A.*
4. *If R is a total ordering and $x \in A$ is a minimal element of A, then x is the smallest element of A. Likewise, if $x \in A$ is a maximal element of A, then x is the largest element of A. In particular, a total ordering has at most one minimal element and one maximal element.*

Item (4) says that we have an even closer connection between minimal and smallest elements under a total ordering.

Minimal and Maximal, VI

3. Let R be a partial ordering on a set A. If $x \in A$ is smallest then x is the unique minimal element of A, and if $x \in A$ is largest then x is the unique maximal element of A.

Proof:

- Recall x is smallest when $x \, R \, a$ for all $a \in A$, and x is minimal when $y \, R \, x$ implies $y = x$.

Minimal and Maximal, VI

3. Let R be a partial ordering on a set A. If $x \in A$ is smallest then x is the unique minimal element of A, and if $x \in A$ is largest then x is the unique maximal element of A.

Proof:

- Recall x is smallest when $x\,R\,a$ for all $a \in A$, and x is minimal when $y\,R\,x$ implies $y = x$.
- First suppose x is a smallest element of A.
- Then x is minimal: if $y\,R\,x$ then since x is smallest we also have $x\,R\,y$, so by antisymmetry we see that $y = x$.
- Additionally, if z is some other minimal element, then since x is smallest we have $x\,R\,z$, but since z is minimal this implies $z = x$: thus, x is the unique minimal element.
- A similar argument (with all of the directions reversed) establishes the corresponding result for largest elements.

Minimal and Maximal, VII

4. If R is a total ordering and $x \in A$ is a minimal element of A, then x is the smallest element of A. Likewise, if $x \in A$ is a maximal element of A, then x is the largest element of A. In particular, a total ordering has at most one minimal element and one maximal element.

Proof:

Minimal and Maximal, VII

4. If R is a total ordering and $x \in A$ is a minimal element of A, then x is the smallest element of A. Likewise, if $x \in A$ is a maximal element of A, then x is the largest element of A. In particular, a total ordering has at most one minimal element and one maximal element.

Proof:

- Suppose R is a total ordering and x is minimal.
- Then for any $y \in A$ we either have $y\,R\,x$ or $x\,R\,y$.
- But since x is minimal, $y\,R\,x$ can only happen when $y = x$. So, if $y \neq x$ we must have $x\,R\,y$, meaning that x is the smallest element of A.
- A similar argument works for maximal and largest elements.
- The last statement then follows immediately from (1): there can be at most one smallest and at most one largest element.

Minimal and Maximal, VIII

We remark that item (4) in the proposition is essentially the converse of item (3) – if x is a unique minimal element of A then x is the smallest element of A – but it has an extra hypothesis: namely, that R is a total ordering.

- In fact this extra hypothesis is necessary, although it is not so easy to write down a counterexample: i.e., a set with a unique minimal element but no smallest element.

Minimal and Maximal, VIII

We remark that item (4) in the proposition is essentially the converse of item (3) – if x is a unique minimal element of A then x is the smallest element of A – but it has an extra hypothesis: namely, that R is a total ordering.

- In fact this extra hypothesis is necessary, although it is not so easy to write down a counterexample: i.e., a set with a unique minimal element but no smallest element.
- One reason for this is that in any finite set with a unique minimal element, that unique minimal element is actually smallest. (This can be shown by induction on the number of elements in the set – see the notes for the details.)
- Thus, any possible counterexample must involve an infinite set.

Minimal and Maximal, IX

Here is a partial ordering on a set that has a unique minimal element but no smallest element:

- Take A to be the set of positive real numbers along with an extra number $\star$ under the usual ordering $\leq$, where we also declare $\star \leq \star$ but otherwise $\star$ is not comparable to any of the positive real numbers.
- You can check that this actually is a partial ordering, if you like.

Minimal and Maximal, IX

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- Take A to be the set of positive real numbers along with an extra number $\star$ under the usual ordering $\leq$, where we also declare $\star \leq \star$ but otherwise $\star$ is not comparable to any of the positive real numbers.
- You can check that this actually is a partial ordering, if you like.
- Then $\star$ is the unique minimal element of A (since $y \leq \star$ is only true when $y = \star$).
- But $\star$ is not the smallest element of A, since for example $\star \not\leq 1$.

Minimal and Maximal, X

To finish our discussion, here are some examples of smallest, largest, minimal, and maximal elements that are of concrete interest.

1. If A is the set of positive common divisors of two positive integers a and b and R is the divisibility relation, the largest element of A under R is the greatest common divisor $\gcd(a, b)$.

Minimal and Maximal, X

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1. If A is the set of positive common divisors of two positive integers a and b and R is the divisibility relation, the largest element of A under R is the greatest common divisor $\gcd(a, b)$.
2. If A is the set of positive common multiples of two positive integers a and b and R is the divisibility relation, the smallest element of A under R is the least common multiple $\operatorname{lcm}(a, b)$.

Minimal and Maximal, XI

3. If $\mathcal{F}$ is the collection of sets that are simultaneously subsets of the sets B and C, and R is the subset relation, the largest element of $\mathcal{F}$ under R is the intersection $B \cap C$.

Minimal and Maximal, XI

3. If $\mathcal{F}$ is the collection of sets that are simultaneously subsets of the sets B and C, and R is the subset relation, the largest element of $\mathcal{F}$ under R is the intersection $B \cap C$.
4. If $\mathcal{F}$ is the collection of sets each containing all of the elements of the two sets B and C, and R is the subset relation, the smallest element of $\mathcal{F}$ under R is the union $B \cup C$.

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4. If $\mathcal{F}$ is the collection of sets each containing all of the elements of the two sets B and C, and R is the subset relation, the smallest element of $\mathcal{F}$ under R is the union $B \cup C$.
5. If A is the set of real numbers of the form x^2 for some $x \in \mathbb{R}$, and R is the relation $\leq$, then the smallest element of A is the number 0.

   We usually express this statement in this simpler form: if $x \in \mathbb{R}$ then $x^2 \geq 0$. This seemingly trivial inequality has very many applications in establishing other inequalities.

Upcoming: Functions

Next class, we start the third major topic in this chapter: functions.

- Our first goal is to explain how to give a formal definition for a function using the language of relations.
- We will then establish some very basic properties of functions that can be interpreted in terms of relations.

Upcoming: Functions

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- Our first goal is to explain how to give a formal definition for a function using the language of relations.
- We will then establish some very basic properties of functions that can be interpreted in terms of relations.
- Next, we discuss function composition, which we can pose again in the language of relations, and establish some basic algebraic properties of composition.
- Then, we will discuss the question of when functions possess an inverse function, along with the closely related notions of when a function is one-to-one and when a function is onto.
- Finally, we will discuss bijections – these are functions that are both one-to-one and onto. This discussion lead us quite naturally into the last major topic in the chapter: cardinality.

Summary

We discussed smallest, largest, minimal, and maximal elements, and gave some examples.

We established some properties of smallest, largest, minimal, and maximal elements.

Next lecture: Functions, properties of functions.