Math 1365 (Intensive Mathematical Reasoning)

Lecture #19 of 35 $\sim$ October 25, 2023

Equivalence Relations

- Equivalence Relations
- Equivalence Classes
- Partitions and Equivalence Relations

This material represents §3.2.1-3.2.2 from the course notes.

Recall

Recall the definition of a relation and what it means for a relation to be reflexive, symmetric, and transitive.

Definition

For sets A and B, we say R is a relation from A to B, written $R : A \to B$, when R is a subset of the Cartesian product $A \times B$.

When $(a, b) \in R$, we write $a \; R \; b$.

Additionally, we say that
R is reflexive when $a \; R \; a$ for all $a \in A$.
R is symmetric when $a \; R \; b$ implies $b \; R \; a$ for all $a, b \in A$.
R is transitive when $a \; R \; b$ and $b \; R \; c$ imply $a \; R \; c$ for all $a, b, c \in A$.

Equivalence Relations, I

We now define the general notion of an equivalence relation:

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The prototypical example of an equivalence relation is equality of elements in a set.

- It is not hard to see that the identity relation on any set A is an equivalence relation.
- In particular, equality of integers, equality of rational numbers, equality of real numbers, and equality of sets are all equivalence relations.

Equivalence Relations, II

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- If m is any positive integer, the mod-m congruence relation $\equiv_m$ on integers is an equivalence relation.
- The biconditional relation $\Leftrightarrow$ on logical propositions is an equivalence relation. (That justifies why we also call it "logical equivalence"!)
- The relation of having the same birthday (on the set of people) is an equivalence relation: everyone has the same birthday as themselves, if P has the same birthday as Q then Q has the same birthday as P, and if P, Q and Q, R have the same birthday, then P, R also have the same birthday.

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- The empty relation on $A = \{1,2,3,4\}$ is not an equivalence relation: although it is symmetric and transitive, it is not reflexive.
- The "differs by at most 1" relation $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : |b - a| \leq 1\}$ on integers is not an equivalence relation: it is reflexive and symmetric, but it is not transitive.

Equivalence Relations, IV

In most settings, it is very common to use a symbol like $\sim$ to represent an equivalence relation rather than the letter R, because the letter R produces expressions that are harder to parse.

- In our discussion, we will continue to use the letter R because we are still examining basic properties of equivalence relations.

Equivalence Classes, I

We saw previously that the residue classes $\overline{a}$ modulo m had a number of very useful properties. There is a natural extension of this concept to a general equivalence relation:

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If R is an equivalence relation on the set A, we define the equivalence class of a as $[a] = \{b \in A : a \, R \, b\}$, the set of all elements $b \in A$ that are related to a via R.

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If R is an equivalence relation on the set A, we define the equivalence class of a as $[a] = \{b \in A : a \, R \, b\}$, the set of all elements $b \in A$ that are related to a via R.

Notice that when R is the mod-m congruence relation on integers, the equivalence class $[a]$ of the element a is the residue class $\overline{a} = \{b \in \mathbb{Z} : a \equiv b \pmod{m}\}$. So the definition above generalizes our earlier notion of a residue class.

Equivalence Classes, II

Example: Let R be the equivalence relation $R = \{(1,1), (1,4), (2,2), (2,3), (3,2), (3,3), (4,1), (4,4)\}$ on $A = \{1,2,3,4\}$. Find the equivalence classes [1], [2], [3], and [4].

Equivalence Classes, II

Example: Let R be the equivalence relation $R = \{(1,1), (1,4), (2,2), (2,3), (3,2), (3,3), (4,1), (4,4)\}$ on $A = \{1,2,3,4\}$. Find the equivalence classes [1], [2], [3], and [4].

- The equivalence classes are $[1] = \{1,4\}$, $[2] = \{2,3\}$, $[3] = \{2,3\}$, and $[4] = \{1,4\}$.
- Notice that there are two different equivalence classes, namely $[1] = [4] = \{1,4\}$ and $[2] = [3] = \{2,3\}$, and also notice that every element of A lies in exactly one of these equivalence classes.

Equivalence Classes, III

Example: Let R be the equivalence relation $R = \{(1,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$ on $A = \{1,2,3,4,5\}$. Find the equivalence classes [1], [2], [3], [4], [5].

Equivalence Classes, III

Example: Let R be the equivalence relation $R = \{(1,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$ on $A = \{1,2,3,4,5\}$. Find the equivalence classes [1], [2], [3], [4], [5].

- The equivalence classes are $[1] = \{1\}$, $[2] = \{2\}$, $[3] = \{3,4\} = [4]$, and $[5] = \{5\}$.
- Notice that there are four different equivalence classes and that every element of A lies in exactly one of these equivalence classes.

Equivalence Classes, IV

Example: If R is the identity relation on the set A, describe the equivalence class $[a]$.

Equivalence Classes, IV

Example: If R is the identity relation on the set A, describe the equivalence class $[a]$.

- This is a bit of a trick, because the equivalence class $[a]$ of the element a is simply the set $\{a\}$ containing a by itself.
- That's because since no other elements of A are related to a except for a itself.

Equivalence Classes, V

Example: Under the equivalence relation "having the same birthday" on the set of people, what is the equivalence class $[P]$ of a person P? How many different equivalence classes are there?

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Example: Under the equivalence relation "having the same birthday" on the set of people, what is the equivalence class $[P]$ of a person P? How many different equivalence classes are there?

- Again by definition, the equivalence class $[P]$ is the set of everyone who has the same birthday as P.
- Notice that we may label these equivalence classes by the shared birthday (e.g., January 1, January 2, ... , up through December 31).
- From this description, we can see that there are exactly 366 equivalence classes (one for each possible birthday, including February 29) and every person lies in exactly one of these equivalence classes (namely, the one labeled with their birthday).

Properties of Equivalence Classes, I

Let's show that some of the properties we saw in the examples hold for general equivalence classes:

Proposition (Properties of Equivalence Classes)

Suppose R is an equivalence relation on the set A. Then

1. *For any $a \in A$, a is an element of $[a]$.*
2. *If $a, b \in A$, then $[a] = [b]$ if and only if $a \, R \, b$.*
3. *Two equivalence classes of R on A are either disjoint or identical.*
4. *There is a unique equivalence class of R on A containing a, namely, $[a]$.*

Properties of Equivalence Classes, II

1. For any $a \in A$, a is an element of $[a]$.

Proof:

Properties of Equivalence Classes, II

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Proof:

- Since R is reflexive, $a \, R \, a$, so by definition, $a \in [a]$.

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2. If $a, b \in A$, then $[a] = [b]$ if and only if $a \, R \, b$.

Proof:

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- Since R is reflexive, $a\ R\ a$, so by definition, $a \in [a]$.

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2. If $a, b \in A$, then $[a] = [b]$ if and only if $a\ R\ b$.

Proof:

- First suppose that $[a] = [b]$.
- By (1), since $b \in [b]$, we see that $b \in [a]$.
- But then by definition of the equivalence class $[a]$, that means $a\ R\ b$, as desired.

Properties of Equivalence Classes, III

2. If $a, b \in A$, then $[a] = [b]$ if and only if $a \, R \, b$.

Proof (continued):

- Now suppose $a \, R \, b$. We must show $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

Properties of Equivalence Classes, III

2. If $a, b \in A$, then $[a] = [b]$ if and only if $a\ R\ b$.

Proof (continued):

- Now suppose $a\ R\ b$. We must show $[a] \subseteq [b]$ and $[b] \subseteq [a]$.
- For the first one, suppose $c \in [a]$.
- Then by definition $a\ R\ c$, so by symmetry $c\ R\ a$.
- Now apply transitivity to $c\ R\ a$ and $a\ R\ b$ to see that $c\ R\ b$, and so by symmetry again, we see $b\ R\ c$.
- Hence by definition, $c \in [b]$, so we conclude $[a] \subseteq [b]$.
- A very similar argument shows the other containment $[b] \subseteq [a]$, and so $[a] = [b]$.

Properties of Equivalence Classes, IV

3. Two equivalence classes of R on A are either disjoint or identical.

Proof:

Properties of Equivalence Classes, IV

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Proof:

- Suppose that $[a]$ and $[b]$ are two equivalence classes of R.
- If they are disjoint, we are done, so suppose there is some c contained in both: then $a\ R\ c$ and also $b\ R\ c$.
- By symmetry, $b\ R\ c$ implies $c\ R\ b$, and then by transitivity, we conclude that $a\ R\ b$.
- Then by property (2), we conclude $[a] = [b]$.
- Hence the two equivalence classes $[a]$ and $[b]$ are either disjoint or identical, as claimed.

Properties of Equivalence Classes, V

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Properties of Equivalence Classes, V

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Proof:

- By (1), clearly $[a]$ is an equivalence class of R containing a.
- On the other hand, by property (3), any other equivalence class containing a must equal $[a]$, so in fact, $[a]$ is the unique equivalence class of R containing a.

Partitions and Equivalence Relations, I

From the results in the proposition, we can see that the equivalence classes are nonempty, pairwise disjoint subsets of A whose union is A. This particular situation is given a name:

Definition

If A is a set, a partition $\mathcal{P}$ of A is a family of nonempty, pairwise disjoint sets whose union is A. The sets in $\mathcal{P}$ are called parts of the partition.

The idea of a partition is simply that it breaks up the set A into smaller nonoverlapping parts.

Partitions and Equivalence Relations, II

Examples:

- The sets $\{1, 5\}$ and $\{2, 3, 4\}$ yield a partition of $\{1, 2, 3, 4, 5\}$; explicitly, we could write $\mathcal{P} = \{\{1, 5\}, \{2, 3, 4\}\}$.

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- The sets $\mathbb{Z}_+ = \{1,2,3,\dots\}$, $\{0\}$, and $\mathbb{Z}_- = \{-1,-2,-3,\dots\}$ yield a partition of the integers.

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- The sets $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$, $\{0\}$, and $\mathbb{Z}_- = \{-1, -2, -3, \dots\}$ yield a partition of the integers.
- The sets $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ and $O = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$ of even and odd numbers yield a different partition of the integers.

Partitions and Equivalence Relations, III

Non-Examples:

- The sets $\{1, 2\}$, $\{3, 4\}$, and $\{4, 5\}$ do not form a partition of $\{1, 2, 3, 4, 5\}$ because the sets are not pairwise disjoint: specifically, $\{3, 4\}$ and $\{4, 5\}$ have the element 4 in common.
- The sets $\{1, 2, 3\}$ and $\{5\}$ do not form a partition of $\{1, 2, 3, 4, 5\}$ because the union of the sets is not all of $\{1, 2, 3, 4, 5\}$: the element 4 is missing.

Partitions and Equivalence Relations, IV

Our results above show that if R is any equivalence relation on a set A, then the equivalence classes of R yield a partition of A. Let's verify this explicitly:

- The union of the equivalence classes is A, because every element of A lies in a residue class.
- Equivalence classes are nonempty, because $a \in [a]$ for any $a \in A$.
- And finally, equivalence classes are pairwise disjoint, because different equivalence classes are disjoint.

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- Equivalence classes are nonempty, because $a \in [a]$ for any $a \in A$.
- And finally, equivalence classes are pairwise disjoint, because different equivalence classes are disjoint.

In fact, the converse of the result above is also true!

- In other words, if we have a partition of A, then it arises as the family of equivalence classes of an equivalence relation on A.

Partitions and Equivalence Relations, V

To illustrate the idea, consider the partition $\mathcal{P} = \{\{1,5\},\{2,3,4\}\}$ of $\{1,2,3,4,5\}$. Let's try to write down an equivalence relation R with those equivalence classes: $\{1,5\}$ and $\{2,3,4\}$.

Partitions and Equivalence Relations, V

To illustrate the idea, consider the partition $\mathcal{P} = \{\{1,5\},\{2,3,4\}\}$ of $\{1,2,3,4,5\}$. Let's try to write down an equivalence relation R with those equivalence classes: $\{1,5\}$ and $\{2,3,4\}$.

- First, R must contain the ordered pairs $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, and $(5,5)$ since it is reflexive.
- Also, R must also contain the pairs $(1,5)$ and $(5,1)$ because 1 and 5 are supposed to lie in the same equivalence class $\{1,5\}$.
- Likewise, R must contain all of the pairs $(2,3)$, $(2,4)$, $(3,2)$, $(3,4)$, $(4,2)$, and $(4,3)$ because 2, 3, and 4 all lie in the same equivalence class.
- Can R contain any other pairs?

Partitions and Equivalence Relations, V

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- Likewise, R must contain all of the pairs $(2,3)$, $(2,4)$, $(3,2)$, $(3,4)$, $(4,2)$, and $(4,3)$ because 2, 3, and 4 all lie in the same equivalence class.
- Can R contain any other pairs? No, because the only pairs left will mix elements from different parts of the partition, and those elements aren't supposed to be related to each other.
- So the only choice is $R = \{(1,1),(1,5),(5,1),(5,5),$ $(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$.

Partitions and Equivalence Relations, VI

We showed $R = \{(1,1),(1,5),(5,1),(5,5),(2,2),(2,3),(2,4),$ $(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$ was the only possible relation whose equivalence classes are the parts $\{1,5\}$ and $\{2,3,4\}$ of the partition $\mathcal{P} = \{\{1,5\},\{2,3,4\}\}$ of $\{1,2,3,4,5\}$.

Is this R actually an equivalence relation?

Partitions and Equivalence Relations, VI

We showed $R = \{(1,1),(1,5),(5,1),(5,5),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$ was the only possible relation whose equivalence classes are the parts $\{1,5\}$ and $\{2,3,4\}$ of the partition $\mathcal{P} = \{\{1,5\},\{2,3,4\}\}$ of $\{1,2,3,4,5\}$.

Is this R actually an equivalence relation? Yes:

- Observe $a\ R\ b$ when both a, b are both in $\{1,5\}$ or in $\{2,3,4\}$.
- So then $a\ R\ a$ because a, a are both in the same part.
- Also, if $a\ R\ b$ then a, b are in the same part, but then so are b, a, meaning $b\ R\ a$.
- And finally, if $a\ R\ b$ and $b\ R\ c$ then a, b and b, c are in the same part, meaning all three of a, b, c are in the same part, so in particular so are a, c, and that means $a\ R\ c$.

Notice R is the union of the Cartesian products $\{1,5\} \times \{1,5\}$ and $\{2,3,4\} \times \{2,3,4\}$ of the underlying parts of the partition.

Partitions and Equivalence Relations, VII

In fact, we can use the same construction idea as in the example to prove the result in general:

Theorem (Equivalence Relations and Partitions)

Let A be a set. If R is any equivalence relation on A, then the equivalence classes of R form a partition $\mathcal{P}$ of A.

Conversely, if $\mathcal{P}$ is a partition of A, then there exists a unique equivalence relation R on A whose equivalence classes are the sets in $\mathcal{P}$, namely, the equivalence relation $R = \bigcup_{X \in \mathcal{P}} (X \times X)$ consisting of all ordered pairs of elements that are in the same part X of the partition $\mathcal{P}$.

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The notation may look a little scary, but it's just a formal way to say that the relation R simply says $a \, R \, b$ precisely when a and b are elements of the same part of the partition $\mathcal{P}$.

Partitions and Equivalence Relations, VIII

Proof (preamble):

- We showed earlier that the equivalence classes of an equivalence relation form a partition.
- Now suppose $\mathcal{P}$ is a partition and consider the relation $R = \bigcup_{X \in \mathcal{P}} X \times X$ consisting of all ordered pairs of elements that are in the same part X of the partition $\mathcal{P}$.
- We need to show the following things:
  1. The relation R is reflexive.
  2. The relation R is symmetric.
  3. The relation R is transitive.
  4. The equivalence classes of R are the parts of the partition $\mathcal{P}$.
  5. R is the unique equivalence relation whose equivalence classes are the parts of $\mathcal{P}$.

Partitions and Equivalence Relations, IX

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

1. The relation R is reflexive.

Proof:

Partitions and Equivalence Relations, IX

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

1. The relation R is reflexive.

Proof:

- For any $a \in A$, by the definition of a partition we must have $a \in X$ for some $X \in \mathcal{P}$.
- But then (a, a) is an element of $X \times X$, as required.

---

2. The relation R is symmetric.

Proof:

Partitions and Equivalence Relations, IX

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

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Proof:

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- But then (a, a) is an element of $X \times X$, as required.

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2. The relation R is symmetric.

Proof:

- Suppose $(a, b) \in R$.
- By the definition of R as a union, we must have $(a, b) \in X \times X$ for some $X \in \mathcal{P}$.
- This means $a \in X$ and $b \in X$.
- But then $(b, a) \in X \times X$ also, so $(b, a) \in R$.

Partitions and Equivalence Relations, X

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

3. The relation R is transitive.

Proof:

Partitions and Equivalence Relations, X

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

3. The relation R is transitive.

Proof:

- Suppose $(a, b) \in R$ and $(b, c) \in R$.
- Then we must have $(a, b) \in X \times X$ and $(b, c) \in Y \times Y$ for some $X, Y \in \mathcal{P}$.
- This means $a \in X$ and $b \in X$, and also $b \in Y$ and $c \in Y$.
- But because $\mathcal{P}$ is a partition, since $b \in X$ and $b \in Y$ we must have $X = Y$.
- Then $a \in X$ and also $c \in X$, so $(a, c) \in X \times X$ and so $(a, c) \in R$.

So now we have shown R is an equivalence relation.

Partitions and Equivalence Relations, XI

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

4. The equivalence classes of R are the parts of the partition $\mathcal{P}$.

Proof:

Partitions and Equivalence Relations, XI

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

4. The equivalence classes of R are the parts of the partition $\mathcal{P}$.

Proof:

- Let $a \in A$ and consider the equivalence class $[a]$ of a.
- Since $\mathcal{P}$ is a partition, $a \in X$ for a unique $X \in \mathcal{P}$. We claim that $[a] = X$.
- To see this, if $b \in X$, we have $(a, b) \in X \times X$ hence $(a, b) \in R$ hence $a \, R \, b$ hence $b \in [a]$. This shows $X \subseteq [a]$.
- For the other containment, if $b \in [a]$ then $a \, R \, b$ so that $(a, b) \in R$.
- By the definition of R as a union, this requires $(a, b) \in Y \times Y$ for some $y \in \mathcal{P}$ where $a \in Y$ and $b \in Y$.
- Since $a \in X$ we must have $Y = X$, so we see $b \in X$. This shows $[a] \subseteq X$, so $[a] = X$ as claimed. We win.

Partitions and Equivalence Relations, XII

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

5. R is the unique equivalence relation whose equivalence classes are the parts of $\mathcal{P}$.

Proof:

Partitions and Equivalence Relations, XII

Let $\mathcal{P}$ be a partition of A, and define $R = \bigcup_{X \in \mathcal{P}} X \times X$.

5. R is the unique equivalence relation whose equivalence classes are the parts of $\mathcal{P}$.

Proof:

- Suppose the equivalence classes of S are the parts of $\mathcal{P}$.
- Then for each $X \in \mathcal{P}$, the relation S must contain $X \times X$, since for any $a, b \in X$, $(a, b) \in S$ because a and b are in the same part of P, hence have the same equivalence class.
- Thus S must contain $R = \bigcup_{X \in \mathcal{P}} X \times X$.
- If S contained any additional ordered pairs, then such an ordered pair would contain elements from two different parts X and Y of the partition.
- But then $X \cup Y$ would be contained in an equivalence class of S, contrary to hypothesis.
- Hence we must have $S = R$, so R is unique as claimed.

Partitions and Equivalence Relations, XIII

The theorem we just proved gives us another way to show that a relation is an equivalence relation: namely, we can check whether it is obtained from a partition.

- This might not sound so useful, but actually, it can save a lot of time.
- All we have to do is identify what the parts of the partition would be, and then we can check to see that all of the necessary pairs are included in the relation R.
- For example $R = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$ is an equivalence relation on $\{1,2,3\}$ because it corresponds to the partition $\{1,3\}$, $\{2\}$.

Summary

We introduced equivalence relations and gave some examples.

We discussed equivalence classes and proved some of their properties.

We discussed the relationship between partitions and equivalence relations.

Next lecture: Construction of $\mathbb{Q}$, partial and total orderings.