Math 1365 (Intensive Mathematical Reasoning)

Lecture #18 of 35 $\sim$ October 23, 2023

Relations, Equivalence Relations

- Relations
- Examples and Properties of Relations
- Reflexive, Symmetric, and Transitive Relations

This material represents §3.1-3.2.1 from the course notes.

Overview of §3

We now begin chapter 3 of the course notes.

- First (today), we will examine the general idea of a relation, which captures the idea of a comparison between two objects.
- Second (this week), we discuss equivalence relations, which generalize the concepts of equality and modular congruence.
- Third (Thursday and next week) we examine orderings, which generalize the "order relations" of subset (on sets), divisibility (on integers), and the ordering on real numbers.
- Fourth (next week and the following week), we construct a formal definition for a function as a special type of relation, and then discuss various properties of functions.
- Finally (about 2-3 weeks from now), we will discuss cardinality – the size of sets – and countability, with a particular focus on the peculiar and exciting properties of infinite sets.

Relations, I

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3. The containment relation $\in$ on an element and a set.

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2. The order relations $\leq$ and $<$ and $\geq$ and $>$ on a pair of integers (or rational numbers, or real numbers).
3. The containment relation $\in$ on an element and a set.
4. The divisibility relation $|$ on a pair of integers.
5. The mod-m congruence relation $\equiv$ on a pair of integers.

Relations, II

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- For example, $5 < 3$ is a statement about the two numbers 5 and 3 (it is a false statement, of course).
- The order of the objects in the relation statement is quite clearly important: for example, $3|6$ is true while $6|3$ is false.
- Also, the objects in a relation statement need not be drawn from the same universe: in the containment relation $x \in A$, for example, the object x can be anything, while the object A is a set.

Relations, III

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In fact, we will take this as the definition of a relation!

Definition

If A and B are sets, we say R is a relation from A to B, written $R : A \to B$, if R is a subset of the Cartesian product $A \times B$.

For any $a \in A$ and $b \in B$, we write $a\ R\ b$ if the ordered pair (a, b) is an element of R, and we write $a \not R\ b$ if the ordered pair (a, b) is not an element of R.

We think of the statement $a\ R\ b$ as saying the ordered pair (a, b) satisfies the relation R.

Relations, IV

We can recast all of the familiar relations we have encountered already in this language of Cartesian products.

<u>Example</u>: Consider the relation $R :\leq$ on integers.

- We can describe R as the set $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b - a \in \mathbb{Z}_{\geq 0}\}$ of ordered pairs (a, b) where $b - a$ is a nonnegative integer.

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- Under this definition, we see that $3 \; R \; 5$ and $4 \; R \; 13$ because $5 - 3 = 2$ and $13 - 4 = 9$ are both nonnegative integers.
- On the other hand, $2 \not R \; 0$ because $0 - 2 = -2$ is not a nonnegative integer.

Relations, V

Example: The divisibility relation $R : |$ on integers can be defined by taking $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : \exists k \in \mathbb{Z} \text{ such that } b = ka\} = \{(a, ka) : a, k \in \mathbb{Z}\}$.

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- Under this definition, we see that $3 \; R \; 6$ and $4 \; R \; 20$ because the ordered pairs $(3, 6) = (3, 2 \cdot 3)$ and $(4, 20) = (4, 5 \cdot 4)$ are in the set described above.
- On the other hand, $2 \not R \; 3$ because $(2, 3)$ is not in the set above.

Relations, VI

Example: The congruence relation $R :\equiv_m$ modulo m can be defined by taking $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \,:\, \exists k \in \mathbb{Z}$ such that $b - a = km\} = \{(a, a+km) \,:\, a,k \in \mathbb{Z}\}$.

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- Under this definition, if $m = 5$ we see that $3 \; R \; 18$ and $4 \; R \; -6$ because the ordered pairs $(3, 18) = (3, 3 + 3 \cdot 5)$ and $(4, -6) = (4, 4 + (-2) \cdot 5)$ are in the set described above.
- On the other hand, $1 \not R \; 3$ because $(1, 3)$ is not in the set above.

Relations, VII

Example: If A is any set, the identity relation is defined by taking $R = \{(a, a) : a \in A\}$. This is simply the equality relation, in which $a \; R \; b$ precisely when a and b are equal.

- Under this definition, if $A = \mathbb{R}$ for example, we see that $3 \; R \; 3$ since $(3, 3)$ is an element of the set R.
- But $1 \not R \; 3$ and $3 \not R \; \pi$ since $(1, 3)$ and $(3, \pi)$ are not elements of R.

Relations, VIII

Here are some other things we can rephrase in the language of relations:

1. The relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : \gcd(a, b) = 1\}$ is the "is relatively prime" relation on integers: we have $a\ R\ b$ precisely when a and b are relatively prime.

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2. The relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y\} = \{(y^2, y) : y \in \mathbb{R}\}$ is the "is a square root of" relation on real numbers: we have $x \, R \, y$ precisely when x is a square root of y (i.e., when $x^2 = y$).

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3. The relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : |b - a| = 1\}$ is the "differs by 1" relation on integers: we have $a \, R \, b$ precisely when a and b differ by 1.

But relations need not have any kind of nice description: *any* arbitrary subset of a Cartesian product $A \times B$ is a relation.

Relations, IX

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7\}$, then some relations are as follows:

- The relation $R_1 = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is a relation from A to B.

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- The relation $R_2 = \{(1, 1), (3, 2), (5, 3), (7, 4)\}$ is a relation from B to A.

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- The relation $R_3 = \{(1, 4), (3, 2), (2, 1)\}$ is a relation from A to A. (We say R_3 is a relation on A.)

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- The relation $R_5 = \{(7, 1), (7, 3)\}$ is a relation from B to A. It is also a relation from B to B.

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- The relation $R_4 = \{(1, 3), (3, 1), (4, 3)\}$ is a relation from A to A. It is also a relation from A to B.
- The relation $R_5 = \{(7, 1), (7, 3)\}$ is a relation from B to A. It is also a relation from B to B.
- The relation $R_6 = \{(1, 1), (3, 3)\}$ is a relation from A to A. It is also a relation from A to B, and from B to A, and from B to B.

Relations, X

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7\}$, then some more relations are as follows:

- The relation $R_7 = \{(1, 1), (2, 7), (3, 5), (5, 4)\}$ is a relation but it is not a relation on A or on B, or from A to B, or from B to A.

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- The relation $R_8 = \{(1, 1),\ (1, 3),\ (1, 7),\ (2, 3), (2, 5), (3, 1), (3, 3), (3, 5), (3, 7), (4, 1), (4, 3), (4, 7)\}$ is a relation from A to B.

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- The relation $R_9 = \{(1, 1), (2, 7), (3, 5), (5, 4)\}$ is not a relation on A or on B, nor is it a relation from A to B or from B to A. (It is a perfectly good relation on $\mathbb{Z}$, however!)

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- The empty relation $R_0 = \{\} = \emptyset$ is a relation from A to A, and also from A to B, and from B to A, and from B to B.

Relations, XI

Since relations are merely subsets of a Cartesian product, we can apply any of our set operations to them.

- For example, if R_1 and R_2 are two relations from A to B, the intersection $R_1 \cap R_2$ is also a relation from A to B.
- Likewise, the union $R_1 \cup R_2$ is also a relation from A to B.

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In a similar way, if C is a subset of A and D is a subset of B, then if $R_{A,B} : A \to B$ is a relation, we may construct a new relation $R_{C,D} : C \to D$ given by $R_{C,D} = R_{A,B} \cap (C \times D)$.

- This relation is called the restriction of R to $C \times D$.
- In the case where R is a relation on A and C is a subset of A, we call $R \cap (C \times C)$ the restriction of R to C, and denote it as $R|_C$.

Relations, XII

Another useful construction is the inverse of a relation, obtained by reversing all of the ordered pairs:

Definition

If $R : A \to B$ is a relation, then the inverse relation (also sometimes called the converse relation or the transpose relation) $R^{-1} : B \to A$ is defined as $R^{-1} = \{(b, a) \,:\, (a, b) \in R\}$, the relation on $B \times A$ consisting of the reverses of all of the ordered pairs in R.

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If $R : A \to B$ is any relation, then it is not hard to see that $(R^{-1})^{-1} = R$, since if $(a, b) \in R$ then $(b, a) \in R^{-1}$ so $(a, b) \in (R^{-1})^{-1}$, and vice versa.

Relations, XIII

Example:

- If $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7\}$, then the inverse of the relation $R_1 = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ from A to B is the relation $R_1^{-1} = \{(1, 1), (3, 2), (5, 3), (7, 4)\}$ from B to A.

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Example:

- If $A = \mathbb{R}$, then the inverse of the relation $R_2 :\leq$ is $R_2^{-1} :\geq$.
- This follows from the observation that $(a, b) \in R_2$ precisely when $b - a$ is nonnegative, and therefore $(b, a) \in R_2^{-1}$ precisely when $b - a$ is nonnegative (which is to say, when the first element of the ordered pair is greater than or equal to the second element).

Relations, XIV

As you can see, the notion of a relation is very general. In fact, it's so general that arbitrary relations are not particularly interesting, since we can't say much about them!

- What we will do now is look at particular kinds of relations with additional properties, and study the structure that these properties impose on relations satisfying them.
- By doing this, we will be able to extend familiar properties of relations like equality and subset to other situations.

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- By doing this, we will be able to extend familiar properties of relations like equality and subset to other situations.

Additionally, in practice, most of the time we do not explicitly work with the definition of a relation as a set of ordered pairs.

- Instead, we think of a relation $a\ R\ b$ as a true or false statement that captures some information about a and b, and we usually work using the language of relation statements $a\ R\ b$ directly, rather than thinking purely in terms of subsets of Cartesian products.

Motivating Equivalence Relations

Let's start by discussing relations that share similar properties to equality.

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- In addition to equality itself (of numbers or of sets), we have already encountered another relation that shares many properties of equality, namely, congruence modulo m.
- The fundamental properties of equality and modular congruence that involve only properties of the relation itself (and not other properties of arithmetic like addition or multiplication) are as follows: for any a, b, c, we have
  (i) $a = a$,
  (ii) if $a = b$ then $b = a$, and
  (iii) if $a = b$ and $b = c$, then $a = c$.
- Let's give general definitions for each of these properties.

Reflexive, Symmetric, Transitive, I

Here are the formalizations, which are simply the corresponding properties of equality but with a general relation R:

Definition

Suppose $R : A \to A$ is a relation on the set A.

We say R is reflexive if $a \; R \; a$ for all $a \in A$.

We say R is symmetric if $a \; R \; b$ implies $b \; R \; a$ for all $a, b \in A$.

We say R is transitive if $a \; R \; b$ and $b \; R \; c$ together imply $a \; R \; c$ for all $a, b, c \in A$.

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We say R is transitive if $a\ R\ b$ and $b\ R\ c$ together imply $a\ R\ c$ for all $a, b, c \in A$.

In formal language:

- R is reflexive when $\forall a \in A,\ a\ R\ a$.
- R is symmetric when $\forall a \in A \forall b \in A,\ (a\ R\ b) \Rightarrow (b\ R\ a)$.
- R is transitive when $\forall a \in A \forall b \in A \forall c \in A,\ [(a\ R\ b) \wedge (b\ R\ c)] \Rightarrow (a\ R\ c)$.

Reflexive, Symmetric, Transitive, II

Reminders of the definitions:

R is reflexive if $a\ R\ a$ for all $a \in A$.

R is symmetric if $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in A$.

R is transitive if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which of the three properties does the identity relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ have?

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Example: Suppose $A = \{1, 2, 3, 4\}$. Which of the three properties does the identity relation $R = \{(1,1), (2,2), (3,3), (4,4)\}$ have?

- Since the relation contains (a, a) for each $a \in A$, it is reflexive.
- If $(a, b) \in R$ then $b = a$, and so $(b, a) = (a, a) \in R$ too, so it is symmetric.
- Finally, if (a, b) and $(b, c) \in R$ then $a = b$ and $b = c$ so $a = c$ and thus R is transitive.
- So this relation has all three properties. That's good, because the relation is actually just the equality relation on A!

Reflexive, Symmetric, Transitive, III

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R is transitive if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which of the three properties does the relation $R = \{(1, 1),\ (2, 3),\ (3, 2)\}$ have?

Reflexive, Symmetric, Transitive, III

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Example: Suppose $A = \{1, 2, 3, 4\}$. Which of the three properties does the relation $R = \{(1, 1), (2, 3), (3, 2)\}$ have?

- The relation is missing $(2, 2)$ (and $(3, 3)$ and $(4, 4)$!) so it is not reflexive.
- We can see just by checking all the pairs that if (a, b) is in R then (b, a) is in R as well, so R is symmetric. (Just flip each pair and check whether the flipped pair is in R.)
- The relation is not transitive because $2\ R_2\ 3$ and $3\ R_2\ 2$, but $2\ \not R_2\ 2$.
- So we see this relation is symmetric, but doesn't have the other two properties.

Reflexive, Symmetric, Transitive, IV

Reminders of the definitions:
R is reflexive if $a \, R \, a$ for all $a \in A$.
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R is transitive if $a \, R \, b$ and $b \, R \, c$ imply $a \, R \, c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does $R = \{(1,1), (1,2), (2,1), (2,2), (2,4), (3,3), (4,2), (4,4)\}$ have?

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R is <u>transitive</u> if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

<u>Example</u>: Suppose $A = \{1, 2, 3, 4\}$. Which properties does
$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 4), (3, 3), (4, 2), (4, 4)\}$ have?

- The relation has all four pairs (a, a) so it is reflexive.
- We can also see that R contains the reverses of all its pairs, so it is symmetric.
- It's harder to see whether R is transitive. But it turns out not to be transitive, because $1\ R\ 2$ and $2\ R\ 4$, but $1 \not R\ 4$.
- So we see this relation is reflexive and symmetric, but not transitive.

Reflexive, Symmetric, Transitive, V

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R is symmetric if $a \; R \; b$ implies $b \; R \; a$ for all $a, b \in A$.
R is transitive if $a \; R \; b$ and $b \; R \; c$ imply $a \; R \; c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which of the three properties does the relation $R = \{(1, 2), (2, 4), (1, 4)\}$ have?

Reflexive, Symmetric, Transitive, V

Reminders of the definitions:
R is reflexive if $a \, R \, a$ for all $a \in A$.
R is symmetric if $a \, R \, b$ implies $b \, R \, a$ for all $a, b \in A$.
R is transitive if $a \, R \, b$ and $b \, R \, c$ imply $a \, R \, c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which of the three properties does the relation $R = \{(1, 2), (2, 4), (1, 4)\}$ have?

- The relation is missing all four pairs (a, a) so it is not reflexive.
- It's also missing all of the flips of its pairs so it is not symmetric either.
- But it is in fact transitive: the only a, b, c for which $a \, R \, b$ and $b \, R \, c$ are both true is $a = 1$, $b = 2$, and $c = 4$, and in such a case we also have $a \, R \, c$.
- So we see this relation is transitive, but not reflexive or symmetric.

Reflexive, Symmetric, Transitive, VI

Reminders of the definitions:

R is reflexive if $a \; R \; a$ for all $a \in A$.

R is symmetric if $a \; R \; b$ implies $b \; R \; a$ for all $a, b \in A$.

R is transitive if $a \; R \; b$ and $b \; R \; c$ imply $a \; R \; c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$ have?

Reflexive, Symmetric, Transitive, VI

Reminders of the definitions:
R is reflexive if $a\ R\ a$ for all $a \in A$.
R is symmetric if $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in A$.
R is transitive if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$ have?

- The relation has all four pairs (a, a) so it is reflexive.
- We can also see that R contains the reverses of none of its pairs, so it is not symmetric.
- It's harder to see whether R is transitive. In this case, R is transitive, though checking it directly is tedious.
- An easier way to see transitivity is to recognize this relation is just the strict inequality relation $<$ on A.

Reflexive, Symmetric, Transitive, VII

Reminders of the definitions:
R is reflexive if $a\ R\ a$ for all $a \in A$.
R is symmetric if $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in A$.
R is transitive if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does $R = \{(1,1), (1,4), (2,2), (2,3), (3,2), (3,3), (4,1), (4,4)\}$ have?

Reflexive, Symmetric, Transitive, VII

Reminders of the definitions:
R is reflexive if $a\ R\ a$ for all $a \in A$.
R is symmetric if $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in A$.
R is transitive if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does $R = \{(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)\}$ have?

- The relation has all four pairs (a, a) so it is reflexive.
- We can also see that R contains the reverses of all of its pairs, so it is symmetric.
- Again, transitivity is harder. In this case, R is transitive.
- One way to see this is to observe that $a\ R\ b$ and $b\ R\ c$ are both true only when a, b, c are either all in $\{1, 4\}$ or all in $\{2, 3\}$. And in that case, the ordered pair (a, c) is also in R.

Reflexive, Symmetric, Transitive, VIII

Reminders of the definitions:
R is reflexive if $a \mathrel{R} a$ for all $a \in A$.
R is symmetric if $a \mathrel{R} b$ implies $b \mathrel{R} a$ for all $a, b \in A$.
R is transitive if $a \mathrel{R} b$ and $b \mathrel{R} c$ imply $a \mathrel{R} c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does the empty relation $R = \{\} = \emptyset$ have?

Reflexive, Symmetric, Transitive, VIII

Reminders of the definitions:
R is reflexive if $a\ R\ a$ for all $a \in A$.
R is symmetric if $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in A$.
R is transitive if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$.

Example: Suppose $A = \{1, 2, 3, 4\}$. Which properties does the empty relation $R = \{\} = \emptyset$ have?

- The empty relation is missing all four pairs (a, a) so it is not reflexive.
- However, the empty relation is symmetric, because the conditional statement "for all $a, b \in A$, if $a\ R\ b$ then $b\ R\ a$" is (vacuously) true because the hypothesis "if $a\ R\ b$" is always false.
- For the same reason, the empty relation is also transitive.

Reflexive, Symmetric, Transitive, IX

Example: The order relation $\leq$ on integers is reflexive and transitive but not symmetric.

Reflexive, Symmetric, Transitive, IX

Example: The order relation $\leq$ on integers is reflexive and transitive but not symmetric.

- Recall that we defined $a \leq b$ to mean that $b - a$ is a nonnegative integer, which is to say, an element of the set $\{0, 1, 2, 3, 4, \ldots\}$.
- Then the relation is reflexive because $a \leq a$ (because $a - a = 0$ is nonnegative), and it is transitive because if $a \leq b$ and $b \leq c$ (meaning that $b - a$ and $c - b$ are nonnegative) then $a \leq c$ (because $(c - b) + (b - a) = c - a$ is nonnegative).
- However, the relation is not symmetric because for example $1 \leq 2$ but $2 \not\leq 1$.

Reflexive, Symmetric, Transitive, X

Example: The implication relation $\Rightarrow$ on logical propositions is reflexive and transitive but not symmetric.

Reflexive, Symmetric, Transitive, X

Example: The implication relation $\Rightarrow$ on logical propositions is reflexive and transitive but not symmetric.

- Explicitly, the relation is reflexive because $P \Rightarrow P$ for any P.
- Likewise, it is transitive because $P \Rightarrow Q$ and $Q \Rightarrow R$ together imply $P \Rightarrow R$ as you showed all the way back on homework 1!
- However, the relation is not symmetric because $P \Rightarrow Q$ is not the same as its converse $Q \Rightarrow P$.

Reflexive, Symmetric, Transitive, X

Example: The implication relation $\Rightarrow$ on logical propositions is reflexive and transitive but not symmetric.

- Explicitly, the relation is reflexive because $P \Rightarrow P$ for any P.
- Likewise, it is transitive because $P \Rightarrow Q$ and $Q \Rightarrow R$ together imply $P \Rightarrow R$ as you showed all the way back on homework 1!
- However, the relation is not symmetric because $P \Rightarrow Q$ is not the same as its converse $Q \Rightarrow P$.

Example: The biconditional relation $\Leftrightarrow$ on logical propositions is reflexive, symmetric, and transitive.

- You can think about why this is true.

Reflexive, Symmetric, Transitive, XI

Example: If m is any positive integer, the mod-m congruence relation $\equiv_m$ on integers is reflexive, symmetric, and transitive.

Reflexive, Symmetric, Transitive, XI

Example: If m is any positive integer, the mod-m congruence relation $\equiv_m$ on integers is reflexive, symmetric, and transitive.

- Recall that we write $a \equiv b \pmod{m}$ when m divides $b - a$.
- For the purposes of our discussion in this chapter, we will abbreviate this statement as $a \equiv_m b$ for consistency with our notation $a\ R\ b$ for relations.
- We have (in fact) already shown that this relation is reflexive, symmetric, and transitive as part of our discussion of properties of congruences.
- Explicitly, we showed $a \equiv a \pmod{m}$, that $a \equiv b \pmod{m}$ implies $b \equiv a \pmod{m}$, and that $a \equiv b$ and $b \equiv c \pmod{m}$ imply $a \equiv c \pmod{m}$. Please refer back to that discussion if you are not sure why these properties are true.

Winding Down

Our goal in discussing the reflexive, symmetric, and transitive properties is to be able to define the general notion of an equivalence relation:

Definition

If R is a relation on the set A, we say R is an equivalence relation when it is reflexive, symmetric, and transitive.

Our goal next time will be to discuss equivalence relations in detail, and show how we can generalize some of the convenient properties of equality and modular congruences.

Summary

We introduced the general notion of a relation from one set to another.

We discussed reflexive, symmetric, and transitive relations and introduced the notion of an equivalence relation.

Next lecture: Equivalence relations, equivalence classes.