Math 1365 (Intensive Mathematical Reasoning)

Lecture #15 of 35 $\sim$ October 12, 2023

Midterm #1 Review

Midterm Topics

The midterm exam covers §1.1-§2.4 of the course notes, representing the material on homeworks 1-5, and more specifically these topics:

- Propositions, Boolean logic
- Conditionals
- Proof techniques (contrapositive, contradiction, etc.)
- Sets, subsets
- Union and intersection
- Cartesian products
- Quantifiers and quantifier logic, indexed sets
- Induction
- Divisibility
- GCDs and LCMs
- Properties of GCDs
- The Euclidean algorithm
- Primes and unique prime factorization
- Applications of prime factorization

Exam Information, I

Some notes on the exam format:

- The exam is held in the regular course classroom. There will be a proctor who will distribute and collect exams, and who will be available to answer questions.
- Please try to arrive at least 5 minutes early for the class period, so that you may get settled and have the full 65 minutes to work.
- You are allowed a 1-page note sheet (double-sided, standard 8.5in-by-11in) on which you may write/type/etc. anything you like.
- You are allowed to use a calculator (any kind) on the exam. No problems will require a calculator.

Exam Information, II

Some notes on the exam format:

- There are approximately 7 pages of material: 1 problem is multiple choice+true/false, and the rest are free response.
- Except when instructed otherwise (e.g., in multiple choice questions), you must show all relevant details and justify all steps with rigorous proofs or clear explicit calculations.
- Correct answers without appropriate work may not receive full credit.

Review Problems

Normally, I would distribute copies of the sheet of review problems in lecture, and then take requests for problems to solve from everyone in attendance at the lecture.

- Here is the link to the review problems, in case you don't have them open already:
  `https://web.northeastern.edu/dummit/teaching_fa23_1365/1365_midterm_1_review_problems.pdf`
- (Note to self: also post this link in the chat!)
- I have randomly selected a few problems that I will go over first, and then I will take requests – please make requests for problems through Zoom chat.

Review Problems, I

[8bei] Find a counterexample to each of the following statements:

(b) If p and q are prime, then $p + q$ is never prime.

(c) If n is an integer, then $n^2 + n + 11$ is always prime.

(i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^4 = x$.

Review Problems, I

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(c) If n is an integer, then $n^2 + n + 11$ is always prime.

(i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^4 = x$.

(b) The sum of two odd primes will always be even and bigger than 2, so we need to add 2 to some other prime. An easy choice is $2 + 3 = 5$.

(c) Small n actually give primes: for example, $n = 1$ gives the prime 13, while $n = 5$ gives the prime 41. But if we take $n = 11$, then all the terms are multiples of 11, meaning that the expression will factor: $11^2 + 11 + 11 = 11 \cdot 13$.

(i) This statement says that every real number x has a real number y such that $y^4 = x$. But fourth powers of real numbers are always ≥ 0, so if we take x to be negative, e.g., $x = -1$, then there is no real y with $y^4 = -1$.

Review Problems, II

[9d] For any sets A, B, C, prove $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

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- Remember $S \backslash T = \{x \in S : x \notin T\}$ = elements in S not in T.
- We want to show each set is a subset of the other.
- Suppose $x \in A \backslash (B \cap C)$: then $x \in A$ and $x \notin (B \cap C)$.
- Note $x \notin (B \cap C) = \neg[x \in B \cap C] = \neg[x \in B \wedge x \in C] = (x \notin B) \vee (x \notin C)$ by de Morgan's laws.
- Now, if $x \notin B$, then since $x \in A$ that means $x \in A \backslash B$. And if $x \notin C$ then since $x \in A$ that means $x \in A \backslash C$.
- So either way we see $x \in (A \backslash B) \cup (A \backslash C)$, as desired.
- For the other containment, suppose $y \in (A \backslash B) \cup (A \backslash C)$.
- If $y \in A \backslash B$ then $y \in A$ and $y \notin B$, so $y \notin B \cap C$ and thus $y \in A \backslash (B \cap C)$.
- If $y \in A \backslash C$ then $y \in A$ and $y \notin C$, so again $y \notin B \cap C$ and thus $y \in A \backslash (B \cap C)$. Either way, $y \in A \backslash (B \cap C)$.

Review Problems, III

[9m] Suppose $b_1 = 3$ and $b_{n+1} = 2b_n - n + 1$ for all $n \geq 1$. Prove that $b_n = 2^n + n$ for every positive integer n.

Review Problems, III

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Review Problems, III

[9m] Suppose $b_1 = 3$ and $b_{n+1} = 2b_n - n + 1$ for all $n \geq 1$. Prove that $b_n = 2^n + n$ for every positive integer n.

- Use induction on n.
- Base case: $n = 1$. We have $b_1 = 3 = 2^1 + 1$ as required.
- Inductive step: Suppose that $b_n = 2^n + n$. [To show: $b_{n+1} = 2^{n+1} + n + 1$.]
- We have

$$\begin{aligned} b_{n+1} &= 2b_n - n + 1 \\ &= 2(2^n + n) - n + 1 \\ &= 2^{n+1} + (n + 1) \end{aligned}$$

as desired. This establishes the inductive step so the result holds for all positive integers n by induction.

Review Problems, IV

[9e] Suppose p is a prime and a is a positive integer. If $p|a^2$, prove that $p|a$.

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- This is a result involving primes and divisibility. We only have one key fact about primes and divisibility, the prime divisibility property: if p is a prime and $p|ab$, then $p|a$ or $p|b$.

Review Problems, IV

[9e] Suppose p is a prime and a is a positive integer. If $p|a^2$, prove that $p|a$.

- This is a result involving primes and divisibility. We only have one key fact about primes and divisibility, the prime divisibility property: if p is a prime and $p|ab$, then $p|a$ or $p|b$.
- In our problem, we are given $p|a^2$ and want to show $p|a$.
- Since $a^2 = a \cdot a$, we can try taking $b = a$ in the prime divisibility property.
- In that situation, it says "if $p|a^2$ then $p|a$ or $p|a$".
- But the two statements in the conclusion ($p|a$ or $p|a$) are the same, so in either case it just says $p|a$.
- So, by the prime divisibility property, if $p|a^2$ then $p|a$. Victory.

Review Problems, V

[9b] Suppose n is an integer. Prove that $2|n$ and $3|n$ if and only if $6|n$.

Review Problems, V

[9b] Suppose n is an integer. Prove that $2|n$ and $3|n$ if and only if $6|n$.

- This is an if-and-only-if statement, so we need to show both implications.
- If $6|n$, meaning $n = 6k$ for some k, then certainly $2|n$ and $3|n$, since $n = 2(3k)$ and $n = 3(2k)$.
- Conversely, suppose $2|n$ and $3|n$.
- Since $2|n$, we know $n = 2a$ for some integer a.
- Then $3|n$ says that $3|(2a)$. Now use the prime divisibility property: since 3 is prime, either 3 divides 2 or 3 divides a.
- Since 3 definitely doesn't divide 2, that means 3 must divide a, so $a = 3b$ for some integer b.
- Then, finally, we see $n = 2a = 6b$ and so $6|n$ as required.

Review Problems, VI

[9p] Prove that if a and b are both odd, then $a^2 + b^2 - 2$ is divisible by 8.

Review Problems, VI

[9p] Prove that if a and b are both odd, then $a^2 + b^2 - 2$ is divisible by 8.

- By definition of odd numbers, there exist integers c and d such that $a = 2c + 1$ and $b = 2d + 1$.
- Then $a^2 + b^2 - 2 = (2c + 1)^2 + (2d + 1)^2 - 2 = 4c^2 + 4c + 4d^2 + 4d = 4(c^2 + c + d^2 + d)$.
- From this we see that $a^2 + b^2 - 2$ is certainly divisible by 4.
- To see the divisibility by 8, notice further that $c^2 + c = c(c + 1)$ is always even, because either c or $c + 1$ must be even (so the product is also even). Likewise, $d^2 + d$ is also even, and so the sum $c^2 + c + d^2 + d$ is even as well.
- Then $a^2 + b^2 - 2$ is 4 times an even number, meaning that it is divisible by 8, as claimed.

Requests for Problems

I will now take requests for problems. Please post problem numbers in the Zoom chat (if you haven't already done so) and I will go through as many of them as we have time for.

- I will make a new slide as we go for each problem with the problem text to make it easier for everyone to follow (we will see how fast I can manage this!).
- I will update the posted slides after the lecture is over with the list of additional problems that we went through.

Requested Problems, 1:35pm Lecture

[10r] Show $25^n + 7$ is a multiple of 8 for every positive integer n.

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[10r] Show $25^n + 7$ is a multiple of 8 for every positive integer n.

- Use induction on n.
- Base case: $n = 1$. We have $25^1 + 7 = 32$, which is a multiple of 8.
- Inductive step: suppose that $25^n + 7$ is a multiple of 8. [To show: $25^{n+1} + 7$ is a multiple of 8.]
- For this observe that $25^{n+1} + 7 = 25(25^n + 7) - 25 \cdot 7 + 7 = 25(25^n + 7) - 24 \cdot 7$.
- Both terms $25(25^n + 7)$ and $24 \cdot 7$ are multiples of 8, so their difference is as well.
- This establishes the inductive step, so the result holds by induction.

Requested Problems, 1:35pm Lecture

Find a counterexample for each statement:
[8f] If $n > 1$ is an integer, then $\sqrt{n}$ is always irrational.
[8e] The sum of two irrational numbers is always irrational.

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Requested Problems, 1:35pm Lecture

Find a counterexample for each statement:
[8f] If $n > 1$ is an integer, then $\sqrt{n}$ is always irrational.
[8e] The sum of two irrational numbers is always irrational.

[8f] We want an integer $n > 1$ such that $\sqrt{n}$ is rational. As we showed in class, $\sqrt{2}$ is irrational, and a similar proof shows $\sqrt{3}$ is irrational. But $\sqrt{4} = 2$ is rational, so $n = 4$ is a counterexample.

[8e] We want two irrational numbers whose sum is rational. We have very few examples of irrational numbers, but as shown in class, $\sqrt{2}$ is irrational. The same proof also shows $-\sqrt{2}$ is irrational. But $\sqrt{2} + (-\sqrt{2}) = 0$ is rational. So these numbers give a counterexample.

Requested Problems, 4:35pm Lecture

[8a] Write, and then prove, the contrapositive of
If a and b are integers, then $3a - 9b \neq 2$.

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If a and b are integers, then $3a - 9b \neq 2$.

- The contrapositive is "If $3a - 9b = 2$ then it is not true that both a and b are integers". Or equivalently, "If $3a - 9b = 2$ then either a or b is not an integer".
- We can show this by contradiction: suppose $3a - 9b = 2$. Then factoring gives $3(a - 3b) = 2$, so if a and b were integers this would say 3 divides 2, but that is impossible.
- Therefore, at least one of a and b is not an integer, as required.

Requested Problems, 4:35pm Lecture

[6b] Find the gcd of $a = 921$ and $b = 177$ and write $\gcd(a, b) = xa + yb$ for some integers x and y.

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[6b] Find the gcd of $a = 921$ and $b = 177$ and write $\gcd(a, b) = xa + yb$ for some integers x and y.

- First we apply the Euclidean algorithm to find the gcd:

$$\begin{aligned} 921 &= 5 \cdot 177 + 36 \\ 177 &= 4 \cdot 36 + 33 \\ 36 &= 1 \cdot 33 + 3 \\ 33 &= 11 \cdot 3 \end{aligned}$$

- Since the last nonzero remainder is 3, that is the gcd. Now we solve for the remainders:

$$\begin{aligned} 36 &= 921 - 5 \cdot 177 \\ 33 &= 177 - 4 \cdot 36 = -4 \cdot 921 + 21 \cdot 177 \\ 3 &= 36 - 1 \cdot 33 = 5 \cdot 921 - 26 \cdot 177 \end{aligned}$$

and so we can take $x = 5$ and $y = -26$.

Summary

We discussed exam logistics.

We did some review problems.

Next lecture: Midterm 1.