E. Dummit's Math 1365 ~ Intensive Mathematical Reasoning, Fall 2023 ~ Homework 9, due Tue Nov 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify whether each of the following sets is (i) finite, (ii) countably infinite, or (iii) uncountably infinite:
   (a) The set $\mathbb{Q}_{>0}$ of positive rational numbers.
   (b) The set $\mathbb{R}$ of real numbers.
   (c) The Cartesian product $\{0,1\} \times \{0,1,2,3,4,5,6,7\}$
   (d) The Cartesian product $\{0,1\} \times \mathbb{Z}$.
   (e) The set of subsets of $\mathbb{Z}$.
   (f) The Cartesian product $\emptyset \times \mathbb{Z}$.
   (g) The Cartesian product $\emptyset \times \mathbb{R}$.
   (h) The set of functions $f : \mathbb{R} \to \mathbb{R}$.
   (i) The Cartesian product $\mathbb{Z} \times \mathbb{Q}$.
   (j) The Cartesian product $\mathbb{Z} \times \mathbb{Q} \times \mathbb{R}$.
   (k) The power set of the power set of the power set of $\{1,2,3,4,5\}$.
   (l) The set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers.

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2. In class, we showed that if A is a finite set and $f : A \to A$ is a function, then f is one-to-one if and only if f is onto. The goal of this problem is for you to show via example that both implications are FALSE in the situation where A is an infinite set.
   (a) Find an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that is one-to-one but not onto.
   (b) Find an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that is onto but not one-to-one.
   (c) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.
   (d) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Suppose $f : A \to B$ and $g : B \to A$ are functions.
   (a) Show that if f is one-to-one and $f \circ g = i_B$ is the identity on B, then $g = f^{-1}$.
   (b) Show that if f is onto and $g \circ f = i_A$ is the identity on A, then $g = f^{-1}$.
   (c) Suppose that $f \circ g = i_B$ but $g \circ f \neq i_A$. Show that f is onto but not one-to-one and g is one-to-one but not onto.

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4. Let p be a prime and a be an integer relatively prime to p. The goal of this problem is to give another proof that $a^p \equiv a \pmod{p}$.
   (a) If S is the set of residue classes modulo p, prove that the function $f : S \to S$ given by $f(\overline{b}) = \overline{a} \cdot \overline{b}$ is a bijection. [Hint: $\overline{a}$ has a multiplicative inverse $\overline{a}^{-1}$ modulo p.]
   (b) Show that $\overline{a} \cdot \overline{2a} \cdot \overline{3a} \cdot \cdots \cdot \overline{(p-1)a} = \overline{1} \cdot \overline{2} \cdot \overline{3} \cdot \cdots \cdot \overline{p-1}$ modulo p. [Hint: Use (a) to show that the two products consist of the same terms, merely rearranged.]
   (c) Prove that $\overline{a}^{p-1} = \overline{1}$ modulo p, and deduce that $a^p \equiv a \pmod{p}$.

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5. The goal of this problem is to provide a different proof that there are infinitely many primes. If S is a set of integers, the characteristic function of S is the function $f : \mathbb{Z} \to \{0,1\}$ defined by $f_S(n) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$. We say a set S has period k when its characteristic function has period k, meaning that $f(n+k) = f(n)$ for all integers n. A set is periodic if it is periodic with some period k.

   (a) For a positive integer p, define $S_p = \{np : n \in \mathbb{Z}\}$. Show that S_p is periodic with period p.

   (b) Show that the complement of a periodic set is periodic.

   (c) Show that the characteristic function of $S \cup T$ is $f_{S \cup T}(n) = \max(f_S(n), f_T(n))$.

   (d) Show that the union of two periodic sets is periodic. [Hint: If S has period a and T has period b, show that $S \cup T$ has period ab.]

   (e) Suppose that there are finitely many prime numbers $\{p_1, p_2, ..., p_n\}$. Show that the complement of $S_{p_1} \cup S_{p_2} \cup ... \cup S_{p_n}$ is the set $\{-1, 1\}$. Explain why this is impossible and conclude that there are infinitely many prime numbers.

   Remark: This approach to proving there are infinitely many primes is an adaptation of a proof of Furstenberg.

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6. The goal of this problem is to give another proof that $\mathbb{Q}$ is countable. Consider the function $f : \mathbb{Q}_+ \to \mathbb{Z}_+$ defined as follows: for $a/b \in \mathbb{Q}$ in lowest terms with prime factorizations $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $b = q_1^{b_1} q_2^{b_2} \cdots q_l^{b_l}$, set $f(a/b) = p_1^{2a_1} p_2^{2a_2} \cdots p_k^{2a_k} q_1^{2b_1 - 1} q_2^{2b_2 - 1} \cdots q_l^{2b_l - 1}$.

   (a) Find $f(7/3)$, $f(9/14)$, $f(1/12)$, $f(1)$, and $f(1000/75)$.

   (b) Show that $f(a/b)$ is a positive integer for every positive rational number a/b.

   (c) Show that f is one-to-one. [Hint: You will need to use the fact that the primes $p_1, \ldots, p_k$ and $q_1, \ldots, q_l$ are all distinct.]

   (d) Find $f^{-1}(12)$, $f^{-1}(180)$, $f^{-1}(2)$, and $f^{-1}(2^4 3^2 5^3 7^3 11^1)$.

   (e) Show that f is onto.

   (f) Deduce that f is a bijection and conclude that $\mathbb{Q}_+$ is countable.

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7. The goal of this problem is to give another proof that the power set of the positive integers $\mathbb{Z}_+$ is uncountable. Let S be the set of infinite base-2 sequences $d_1 d_2 d_3 d_4 \ldots$, where each digit $d_i \in \{0, 1\}$ for all $i \geq 1$.

   (a) Prove that S is uncountable. [Hint: Use Cantor's diagonal argument.]

   (b) Show that the function $f : S \to \mathcal{P}(\mathbb{Z}_+)$ given by $f(d_1 d_2 d_3 d_4 \ldots) = \{n : d_n = 1\}$ is a bijection. Deduce that $\mathcal{P}(\mathbb{Z}_+)$ is uncountable.
