E. Dummit's Math 1365 ∼ Intensive Mathematical Reasoning, Fall 2023 ∼ Homework 8, due Tue Nov 7th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each partial ordering on each set S, identify all minimal, maximal, smallest, and largest elements:
   (a) The divisibility ordering on the set $S = \{3, 4, 5, 6, 7, 8, 9\}$.
   (b) The divisibility ordering on the set $S = \{1, 2, 4, 8, 16, 32\}$.
   (c) The divisibility ordering on the set $S = \{-100, -10, -1, 1, 10, 100, 1000000\}$.
   (d) The ordering $\leq$ on the set of real numbers $S = \{x \in \mathbb{R} : 0 < x \leq 2\}$.
   (e) The ordering on $S = \{1, 2, 3\} \times \{1, 2, 3\}$ where we say $(a, b) \leq (c, d)$ if and only if either $(a, b) = (c, d)$ or both $a < c$ and $b < d$.

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2. For each f, A, and B, identify whether or not f is a function from A to B, and if not, briefly explain why not.
   (a) $A = \{1, 2, 3\}$, $B = \{4\}$, where $f = \{(1, 4), (2, 4), (3, 4)\}$.
   (b) $A = \{1\}$, $B = \{2, 3, 4\}$, where $f = \{(1, 2), (1, 3), (1, 4)\}$.
   (c) $A = \{1, 2, 3\}$, $B = \{4\}$, where $f = \{(1, 2), (2, 3), (3, 4)\}$.
   (d) $A = \mathbb{Q}$, $B = \mathbb{Q}$, where $f(a/b) = a/b^2$.
   (e) $A = \mathbb{Q}$, $B = \mathbb{Q}$, where $f(a/b) = a^2/b^2$.
   (f) $A = \mathbb{Z}$, $B = \mathbb{Z}/m\mathbb{Z}$, where $f(a) = \overline{a}$, with m a fixed positive integer.
   (g) $A = \mathbb{Z}/m\mathbb{Z}$, $B = \mathbb{Z}$, where $f(\overline{a}) = a$, with m a fixed positive integer.
   (h) $A = \mathbb{R}$, $B = \mathbb{Z}$, where $f = \{(x, n) \in \mathbb{R} \times \mathbb{Z} : n \leq x < n + 1\}$.

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3. For each function $f : A \to B$, determine whether f is (i) one-to-one, (ii) onto, and (iii) a bijection.
   (a) $f_1(x) = 2x + 1$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
   (b) $f_2(n) = 2n + 1$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.
   (c) $f_3(x) = x^2$ from $A = \mathbb{R}_+$ to $B = \mathbb{R}_+$.
   (d) $f_4(x) = \dfrac{2x - 1}{x + 3}$ from $A = \mathbb{R}\backslash\{-3\}$ to $B = \mathbb{R}$.
   (e) $f_5(n) = \dfrac{1}{n^2 + 1}$ from $A = \mathbb{Z}$ to $B = \mathbb{Q}$.
   (f) $f_6(a) = \overline{a}$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}/m\mathbb{Z}$, where m is a fixed positive integer.
   (g) $f_7(a, b) = (2a + b, 3a)$ from $A = \mathbb{Z} \times \mathbb{Z}$ to $B = \mathbb{Z} \times \mathbb{Z}$.
   (h) $f_8(a, b) = (2a + b, 3a)$ from $A = \mathbb{R} \times \mathbb{R}$ to $B = \mathbb{R} \times \mathbb{R}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. Let A be a set containing exactly n elements for a positive integer n, and let R be a total ordering on A.
   (a) Prove that A contains a largest element. [Hint: Induct on the number of elements n.]
   (b) Prove that it is possible to label the elements of A as $\{a_1, a_2, \ldots, a_n\}$ such that $a_i \, R \, a_j$ precisely when $i \le j$. [Hint: Use (a) to identify a largest element, and then apply induction to the rest of A.]
   (c) Demonstrate the result of part (b) by identifying the labels a_i for the divisibility ordering on the set $\{2, -12, -6, 36, 144\}$.

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5. Suppose $f : A \to B$ is a function, and S is an equivalence relation on B. Prove that the relation $R : A \to A$ given by $R = \{(a, b) \in A \times A \, : \, (f(a), f(b)) \in S\}$ is an equivalence relation on A.

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6. Suppose A, B, and C are sets.
   (a) If $f : B \to C$ and $g : A \to B$ are both one-to-one, prove that $f \circ g$ is also one-to-one.
   (b) If $f : B \to C$ and $g : A \to B$ are both onto, prove that $f \circ g$ is also onto.

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7. Suppose $f : A \to B$ is a function.
   - If $S \subseteq A$, we write $f(S) = \{f(s) : s \in S\}$ and call $f(S)$ the <u>image</u> of S.
   - If $T \subseteq B$, we write $f^{-1}(T) = \{a \in A : f(a) \in T\}$ and call $f^{-1}(T)$ the <u>inverse image</u> of T.
   - When $T = \{b\}$ is a single element, we write $f^{-1}(T)$ as $f^{-1}(b)$ rather than $f^{-1}(\{b\})$, with the understanding that $f^{-1}(b)$ is a set that could be empty or contain more than one element.

   (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is the function with $f(x) = x^2$ and recall the notation $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$ for a closed interval. Match the following ten image or inverse image sets with their values.
   <u>Sets</u>: $f(\{1, 2\})$, $f([-1, 2])$, $f([0, 1])$ $f(\emptyset)$, $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(\{1, 4\})$, $f^{-1}(-1)$, $f^{-1}([4, 9])$, $f^{-1}([0, 1])$.
   <u>Values</u>: $\emptyset$ (twice), $\{-1, 1\}$, $[0, 4]$, $\{0\}$, $[-1, 1]$, $\{-2, -1, 1, 2\}$, $[0, 1]$, $[-3, -2] \cup [2, 3]$, $\{1, 4\}$.
   (b) Suppose $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = \sin(x)$. Find $g^{-1}(0)$, $g^{-1}(2)$, and $g^{-1}([-1, 1])$.
   (c) Suppose $f : A \to B$. If S is any subset of A, show that $S \subseteq f^{-1}(f(S))$.
   (d) Find an example of a subset S of $\mathbb{R}$ such that $S \ne f^{-1}(f(S))$ for the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.
   (e) Suppose $f : A \to B$. If T is any subset of B, show that $f(f^{-1}(T)) \subseteq T$.
   (f) Find an example of a subset T of $\mathbb{R}$ such that $T \ne f(f^{-1}(T))$ for the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.
   (g) Suppose $f : A \to B$. If B_1 and B_2 are subsets of B, show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
   (h) Find an example of subsets A_1 and A_2 of $\mathbb{R}$ such that $f(A_1 \cap A_2) \ne f(A_1) \cap f(A_2)$ for the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.
