E. Dummit's Math 1365 ~ Intensive Mathematical Reasoning, Fall 2023 ~ Homework 5, due Tue Oct 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the following:
   (a) Find the gcd and lcm of 144 and 300.
   (b) Find the gcd and lcm of $2^8 3^{11} 5^7 7^8 11^2$ and $2^4 3^8 5^7 7^7 11^{11}$.
   (c) Find the prime factorizations of 1600, 2023, and 2023^{2024}.

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2. For each pair of integers (a, b), use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ AND also to find integers x and y such that $d = ax + by$. (Make sure to include the Euclidean algorithm calculations in your writeup.)
   (a) $a = 12$, $b = 44$.
   (b) $a = 20$, $b = 107$.
   (c) $a = 2023$, $b = 20234$.
   (d) $a = 5567$, $b = 12445$.
   (e) $a = 233$, $b = 144$.

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3. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:
   (a) Proposition: All horses are the same color.
   Proof: Induction on n, the number of horses. The base case $n = 1$ is trivial because any 1 horse is the same color as itself. For the inductive step, suppose that any $n + 1$ horses are the same color. Ignoring the last horse yields means that we need to show that n horses are the same color, which is true by the induction hypothesis. Therefore the result holds by induction.
   (b) Proposition: For every positive integer n, $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.
   Proof: Induction on n. The base case $n = 1$ follows because $1 = \frac{1}{2}(1)(2)$. To show the inductive step, we want $1 + 2 + 3 + \cdots + n + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$. Subtracting $n + 1$ from both sides yields $1 + 2 + 3 + \cdots + n = \frac{1}{2}(n + 1)(n + 2) - (n + 1) = \frac{1}{2}n(n + 1)$ which is true by the induction hypothesis. Therefore the result holds by induction.
   (c) Proposition: The square root of 4 is irrational.
   Proof: Suppose that $\sqrt{4} = a/b$ is rational and in lowest terms, so that a and b are relatively prime positive integers. Squaring both sides and clearing denominators yields $4b^2 = a^2$. Since $4|a^2$ this means $2|a$ so $a = 2c$ for some integer c. Plugging in and cancelling yields $b^2 = c^2$ so since b and c are positive, we have $b = c$, so $a = 2b$. But this is a contradiction since a and b were assumed to be relatively prime.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. The goal of this problem is to study which numbers of the form $N = a^k - 1$ can be prime, where a and k are positive integers greater than 1.
   (a) Show that $x^k - 1$ is divisible by $x - 1$, for any integer x.
   (b) Show that if $a > 2$, then $N = a^k - 1$ is not prime.
   (c) Show that if k is composite, then $N = 2^k - 1$ is not prime. [Hint: If $k = rs$, show $2^r - 1$ divides N.]
   (d) Conclude that if $N = a^k - 1$ is prime, then $a = 2$ and k is prime. (Such primes are called Mersenne primes.)
   (e) If p is prime, is $2^p - 1$ always prime?

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5. The goal of this problem is to demonstrate that the uniqueness of prime factorizations is not as obvious as it may seem. Let S be a nonempty set of positive integers, and define an S-prime to be an element $p \in S$ such that $p > 1$ and there do not exist $a, b \in S$ such that $ab = p$ and $1 < a, b < p$. (If S is the set of all positive integers, then this definition reduces to the usual one for prime numbers.) Let $E = \{2, 4, 6, 8, 10, \dots\}$ be the set of even positive integers and $O = \{1, 3, 5, 7, 9, 11, \dots\}$ be the set of odd positive integers.

   (a) Which of 2, 4, 6, 8, 10, 12, 14, and 16 are E-primes?

   (b) Show that $2n \in E$ is an E-prime if and only if n is odd. [Hint: Show the contrapositive.]

   (c) Show that 60 has two different factorizations as a product of E-primes. Deduce that E does not have unique E-prime factorization.

   (d) Which of 1, 3, 5, 7, 9, 11, 13, and 15 are O-primes?

   (e) Show that $p \in O$ is an O-prime if and only if p is an odd prime integer.

   (f) Explain why O has unique O-prime factorization.

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6. The Fibonacci-Virahanka numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci-Virahanka sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n. [Hint: Use induction.]

   (b) Prove that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer n.

   (c) Prove that $F_{n+3} - F_n$ is even for every positive integer n.

   (d) A "lyrical pattern" consists of a sequence of long and short beats, where a long beat is twice as long as a short beat. Some examples are long-long-short-long (length 7) and short-short-short-short-long (length 6). Prove that for all $n \geq 1$, the number of lyrical patterns whose length equals n short beats is the Fibonacci number F_{n+1}. [Hint: What happens if you delete the last beat in a sequence of length n?]

   Remark: The study of lyrical patterns by Indian poets writing in Sanskrit (e.g., Pingala in approximately 200 BCE) is the first known analysis of the Fibonacci-Virahanka numbers (historically called the Fibonacci numbers following Fibonacci's description of them in 1202 CE, but Virahanka was the first to give a clear description of them in approximately the year 700 CE). There are very many identities involving the Fibonacci numbers, and they show up in many applications.

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7. Prove that $\log_3 5$ is irrational. [Hint: Suppose otherwise, so that $\log_3 5 = a/b$. Convert this to statement about positive integers and find a contradiction.]
