

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Let  $U = \mathbb{R}_{\geq 0}$ , the nonnegative real numbers, and take  $P(x, y)$  to be the statement " $x < y$ ".
    - (a) There are 8 possible ways, listed below, of quantifying both  $x$  and  $y$  in the statement  $P(x, y)$ . For each statement, translate it into words and then find its truth value.

i. $\forall x \forall y, P(x, y)$	v. $\forall y \forall x, P(x, y)$
ii. $\forall x \exists y, P(x, y)$	vi. $\forall y \exists x, P(x, y)$
iii. $\exists x \forall y, P(x, y)$	vii. $\exists y \forall x, P(x, y)$
iv. $\exists x \exists y, P(x, y)$	viii. $\exists y \exists x, P(x, y)$
    - (b) Repeat part (a) with  $P(x, y)$  replaced by the statement " $y = 2x$ " and with universe  $\mathbb{Z}$ .
    - (c) Of the eight quantified statements in part (a), two pairs will always be logically equivalent for any statement  $P(x, y)$ . Identify these pairs.
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2. For each statement, translate it into words and then find its truth value. (Assume that all capital-letter variables refer to sets.)
    - (a)  $\forall x \in \mathbb{R}, x^2 > 0$ .
    - (b)  $\exists x \in \mathbb{Z}, x^2 - 3x + 2 = 0$ .
    - (c)  $\forall A \forall B \forall C, [x \in A \cap B \cap C] \Rightarrow [x \in A \cap B] \wedge [x \in A \cup C]$ .
    - (d)  $\forall A \exists x (x \in A)$ .
    - (e)  $\forall A \exists x \exists y, (A = \emptyset) \vee [(x \in A) \wedge (y \in A)]$ .
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3. Negate each given statement and then rewrite the result as an equivalent positive statement. (All quantifiers should appear ahead of any negation operators.)
    - (a)  $\exists x, x^2 = 2$ .
    - (b)  $\exists x \exists y, x + y \neq 5$ .
    - (c)  $\forall x \exists y \exists z, x \cdot y + z > 2$ .
    - (d)  $\forall a \in A \exists b \in B (a \in C \wedge b \in C)$ .
    - (e) There exists an  $x$  such that for all  $y$ , it is true that  $y^2 > x$ .
    - (f) For all  $n$  there exist  $a, b$ , and  $c$  such that  $n = a^2 + b^2 + c^2$ .
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4. Suppose that all of Pinocchio's statements are false and that Pinocchio says "All my hats are green". For each statement, identify whether it must be true, must be false, or could be either true or false:
    - (a) Pinocchio has at least one green hat.
    - (b) Pinocchio has no green hats.
    - (c) Pinocchio has no hats.
    - (d) Pinocchio has at least one hat.
    - (e) Pinocchio has exactly one green hat.
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5. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

- (a) Proposition: For all real numbers  $x$ , it is true that  $x^2 \geq 0$ .  
Proof: Suppose by way of contradiction that the desired result is false. Then for all real numbers  $x$ , it is true that  $x^2 < 0$ . But this is statement is incorrect, because taking  $x = 2$  yields the false statement  $4 < 0$ . This is a contradiction, so it must be true that for all real numbers  $x$ ,  $x^2 \geq 0$ .
- (b) Proposition: For all real numbers  $x$  there exists a real number  $y$  such that  $x - 2y = y^2 + 1$ .  
Proof: Let  $x = (y + 1)^2$ : then  $x - 2y = (y + 1)^2 - 2y = (y^2 + 2y + 1) - 2y = y^2 + 1$ , as required.
- (c) Proposition: There exists a real number  $y$  such that for all real numbers  $x$ , it is true that  $x - 2y = y^2 + 1$ .  
Proof: Let  $x = (y + 1)^2$ : then  $x - 2y = (y + 1)^2 - 2y = (y^2 + 2y + 1) - 2y = y^2 + 1$ , as required.
- (d) Proposition: For all real numbers  $x$  and  $y$ , it is true that  $x^2 + 2xy - 3y^2 = 0$ .  
Proof: Suppose  $x$  and  $y$  are arbitrary, say, equal to some number  $r$ . Then  $x^2 + 2xy - 3y^2 = r^2 + 2r^2 - 3r^2 = 0$ . Since  $x$  and  $y$  were both chosen arbitrarily, we conclude that for all real numbers  $x$  and  $y$ , it is true that  $x^2 + 2xy - 3y^2 = 0$ .
- (e) Proposition: For any nonempty sets  $A$  and  $B$ , it is true that  $A \cap B$  is nonempty.  
Proof: By definition since  $A$  is nonempty,  $\exists x, x \in A$ . Likewise, since  $B$  is nonempty,  $\exists x, x \in B$ . But then this  $x$  has  $x \in A$  and  $x \in B$  so by definition of  $A \cap B$  we have  $x \in A \cap B$ , so  $A \cap B$  is nonempty.

**Part II:** Solve the following problems. Justify all answers with rigorous, clear arguments.

6. Many of our proofs involving sets have implicitly used quantifiers. The goal of this problem is to make some of these ideas more explicit by analyzing the proof (given in class) that if  $A$  and  $B$  are any sets then  $A \cap B = A$  is logically equivalent to  $A \subseteq B$ .

- (a) If  $P$  and  $Q$  are propositions, show that  $P \Rightarrow (P \wedge Q)$  is logically equivalent to  $P \Rightarrow Q$ .
- (b) If  $A$  and  $B$  are sets, show that  $\forall x, (x \in A) \Rightarrow [(x \in A) \wedge (x \in B)]$  is logically equivalent to  $\forall x, (x \in A) \Rightarrow (x \in B)$ . [Hint: Use (a).]
- (c) Explain why (b) says that  $A \subseteq A \cap B$  is equivalent to  $A \subseteq B$ .

7. Suppose  $A$  and  $B$  are sets. The goal of this problem is to study the question of when  $A \times B = B \times A$ .

Proposition:  $A \times B = B \times A$  if and only if  $A = B$ .

Proof: If  $A = B$ , then clearly  $A \times B = A \times A = B \times A$ . Now suppose  $A \times B = B \times A$ , and let  $a \in A$  and  $b \in B$  be arbitrary elements of  $A$  and  $B$  respectively. Then by definition,  $(a, b) \in A \times B$ , and so by hypothesis,  $(a, b) \in B \times A$ . This means  $a \in B$  and  $b \in A$ . Since  $a \in A$  and  $b \in B$  are arbitrary, the fact that  $a \in B$  implies  $A \subseteq B$ , and the fact that  $b \in A$  implies  $B \subseteq A$ . We conclude that  $A = B$ , as required.

- (a) Consider the proposition and proof given above. Show that the proposition is incorrect by explaining why taking  $A = \emptyset$  and  $B = \{1, 2\}$  yields a counterexample.
- (b) Part (a) shows that the proposition stated above is incorrect, so the proof must contain a logical error. Identify what the error is, and why it causes the proof to be incorrect. [Hint: The counterexample from part (a) is clearly relevant.]
- (c) Give a corrected version of the proposition, and then give a correct proof. [Hint: Your corrected proposition should start with “ $A \times B = B \times A$  if and only if  $A = B$  or...” and the proof should also make sure to address the error you identified in part (b).]

8. The goal of this problem is to examine the quantifier “there exists a unique”, written as  $\exists!$ . Thus, for example, the statement “there exists a unique  $x$  such that  $x^2 = 2$ ” would be written  $\exists!x, x^2 = 2$ . The meaning of this quantifier is that there exists an element  $x$  satisfying the hypotheses, and that there is exactly one such  $x$ .

- (a) Identify the truth values of the following statements:
  - i.  $\exists!n \in \mathbb{Z}, n^2 = 2$ .    ii.  $\exists!n \in \mathbb{Z}, n^2 = 4$ .    iii.  $\exists!n \in \mathbb{Z}, n^2 = 0$ .    iv.  $\forall x \exists!y, xy = 1$ , with universe  $\mathbb{R}_{\geq 0}$ .
- (b) It may seem that  $\exists!$  is a new quantifier, but in fact, it can be expressed in terms of  $\exists$  and  $\forall$ . Explain why  $\exists!x \in A, P(x)$  is logically equivalent to  $\exists x \in A, P(x) \wedge [\forall y \in A, P(y) \Rightarrow (y = x)]$  for any proposition  $P(x)$ . (Your explanation does not have to be fully rigorous.)