E. Dummit's Math 1365 ∼ Intensive Mathematical Reasoning, Fall 2023 ∼ Homework 11, due Thu Dec 7th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each set G with the given operation, identify whether or not G is a group under the operation:
   (a) The positive real numbers under addition.
   (b) The positive real numbers under multiplication.
   (c) The set of all residue classes $\{\overline{0}, \overline{1}, \dots, \overline{m-1}\}$ modulo m, under multiplication.
   (d) The set of invertible residue classes modulo m, under multiplication.
   (e) The set of functions $f : \mathbb{R} \to \mathbb{R}$ under pointwise addition: $(f + g)(x) = f(x) + g(x)$ for each real x.
   (f) The set of functions $f : \mathbb{R} \to \mathbb{R}$ under function composition: $(f \circ g)(x) = f(g(x))$ for each real x.
   (g) The set of bijections $f : \mathbb{R} \to \mathbb{R}$ under function composition: $(f \circ g)(x) = f(g(x))$ for each real x.

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2. Compute / find each of the following:
   (a) The product $(sr^2)(r^5)$ in the dihedral group $D_{2\cdot 5}$.
   (b) The product $(sr^2)(sr^3)(r^2)(s)$ in the dihedral group $D_{2\cdot 6}$.
   (c) The cycle decomposition of the permutation $\sigma \in S_8$ with $\sigma(1) = 4$, $\sigma(2) = 8$, $\sigma(3) = 5$, $\sigma(4) = 3$, $\sigma(5) = 2$, $\sigma(6) = 7$, $\sigma(7) = 6$, and $\sigma(8) = 1$.
   (d) The cycle decomposition of the product $(3\,4\,5) \cdot (4\,2\,1)$ in S_5.
   (e) The cycle decomposition of the product $(1\,3\,2\,5) \cdot (3\,6) \cdot (1\,6\,4)$ in S_7.
   (f) The cycle decomposition of the inverse of $(1\,5)(2\,6\,3\,4\,7)$ in S_7.
   (g) The 2023rd power of the element $(4\,9)$ in S_{10}.
   (h) The 2nd, 3rd, 4th, 5th, and 2023rd powers of the element $(1\,3\,2\,8\,5)$ in S_{10}.
   (i) The 2023rd power of the element $(1\,3\,2\,8\,5)(4\,9)(6\,7\,10)$ in S_{10}.
   (j) The order of the residue class $\overline{3}$ in the group of residue classes modulo 12 under addition.
   (k) The order of the residue class $\overline{2}$ in the group of invertible residue classes modulo 5 under multiplication.
   (l) The order of the residue class $\overline{5}$ in the group of invertible residue classes modulo 11 under multiplication.
   (m) All possible orders of a subgroup of G, if G has order 30.
   (n) An abelian group of order 6.
   (o) A non-abelian group of order 6.
   (p) A non-abelian group of order 24.
   (q) The least upper bound in $\mathbb{R}$ of the set of rational numbers $\{0.9, 0.99, 0.999, 0.9999, 0, 99999, \dots\}$.
   (r) The least upper bound in $\mathbb{R}$ of the set of rational numbers $\{a/b : a, b \in \mathbb{Z}_+ \text{ and } a^2 \le 3b^2\}$.

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3. As noted in class, the order of any element in the symmetric group S_n is the least common multiple of the lengths of its cycles.

   Example: The element $(1\,2\,3\,4)(5\,6\,7\,8\,9\,10)$ in S_{10} has order $\text{lcm}(4, 6) = 12$.

   (a) What are the 6 possible orders of an element of S_5?
   (b) Find an example of an element of S_5 of each possible order.
   (c) Find an example of an element of S_{10} of order 20.

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4. Consider the dihedral group $D_{2\cdot 5}$ of order 10.

   (a) Identify the orders of each of the 10 elements e, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4 of $D_{2\cdot 5}$.

   (b) Find the possible orders of a subgroup of $D_{2\cdot 5}$, and give a subgroup of each possible order.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Prove that the intersection $S = \bigcap_{i \in I} G_i$ of an arbitrary collection of subgroups G_i of G is a subgroup of G.

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6. Suppose G is a group with the property that $g^2 = e$ for every $g \in G$. Prove that G is abelian.

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7. Let G be a group and let H be a subgroup of G. Prove that the relation R on G defined by $g_1 \, R \, g_2$ when g_1 and g_2 are in the same left H-coset (i.e., when $g_1 H = g_2 H$, which is to say, $g_1 = g_2 h$ for some $h \in H$) is an equivalence relation.

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8. Suppose G is a group. Recall that if $g \in G$ then the subgroup generated by g, denoted $\langle g \rangle$, is the set of all powers of g: $\langle g \rangle = \{\dots, g^{-3}, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots\}$. We say that G is <u>cyclic</u> if there exists an element $g \in G$ such that $\langle g \rangle = G$, and we call such an element g a <u>generator</u> of G.

   Example: The group $(\{1, -1\}, \cdot)$ is cyclic and generated by the element -1.

   Example: The group $(\mathbb{Z}, +)$ is cyclic and generated by the additive identity 1.

   (a) Show that the group $(\mathbb{Z}/m\mathbb{Z}, +)$ is cyclic by identifying a generator.

   (b) Show that if G has order n, then G is cyclic if and only if G contains an element of order n.

   (c) Show that the group $G = \{1, i, -1, -i\}$ under multiplication is cyclic, where $i = \sqrt{-1}$.

   (d) Show that the Klein 4-group $V_4 = \{e, a, b, ab\}$ with $a^2 = b^2 = (ab)^2 = e$ is not cyclic.

   (e) Show that every group of prime order p is cyclic. [Hint: What orders are possible for a non-identity element?]
