Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify whether each of the following sets is finite, countably infinite, or uncountably infinite:
   (a) The set of real numbers that can be described using at most 60 characters written in English. (For example, 1 can be described using the string "one" while π can be described as "the ratio of a circle's circumference to its diameter".)
   (b) The set of real numbers that can be described using at most 100,000 characters written in English.
   (c) The set of real numbers that can be described using a finite number of characters written in English.
   (d) The set of real numbers that <u>cannot</u> be described by any finite string of characters written in English.

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2. Solve the following counting problems:
   (a) Find the number of functions $f : \{a, b, c, d, e\} \to \{1, 2, 3, 4, 5\}$. How many are one-to-one? How many are onto?
   (b) Find the number of functions $f : \{1, 2, 3\} \to \{a, b, c, d, e\}$. How many are one-to-one? How many are onto?
   (c) Find the number of functions $f : \{a, b, c, d, e\} \to \{1, 2, 3\}$. How many are one-to-one? How many are onto? [Hint: For onto functions, try counting how many have each possible image set.]

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Suppose A, B, and C are finite sets. Recall the union-intersection formula $\#(A \cup B) = \#A + \#B - \#(A \cap B)$.
   (a) Show that $\#(A \cup B \cup C) = \#A + \#B + \#C - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C)$. [Hint: Apply the union-intersection formula to $(A \cup B) \cup C$.]
   (b) Find the number of integers in the set $\{1, 2, 3, \ldots, 2023\}$ that are divisible by 2 or by 3 or by 5.

   Remark: There is a natural generalization of (a), the <u>inclusion-exclusion formula</u>, for $\#(A_1 \cup A_2 \cup \cdots \cup A_n)$ in terms of the cardinalities of the various possible intersections of A_1, A_2, ... , A_n, which can be obtained using an induction argument.

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4. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a function such that $f(f(f(n)) = n$ for all $n \in \mathbb{Z}$.
   (a) Show that f is a bijection.
   (b) Give an example of such a function f that is NOT equal to the identity function. (You don't need to give an explicit formula, but at least describe how to find the values of f.)

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5. The goal of this problem is to give another proof that $\mathbb{R}$ is uncountable. Let $f : \mathcal{P}(\mathbb{Z}_+) \to \mathbb{R}$ be the function defined as follows: $f(A)$ is the decimal whose nth decimal place is 1 if $n \in A$, and is 2 if $n \notin A$.
   (a) Find the decimal expansion to 10 digits of the value of f on each of these sets: (i) the set of even integers, (ii) the set of odd integers, and (iii) the set of prime numbers.
   (b) Prove that f is one-to-one. [Hint: Use the fact that $f(A)$ has a unique decimal expansion for any set A.]
   (c) Using the fact that $\mathcal{P}(\mathbb{Z}_+)$ is uncountable, show that $\mathbb{R}$ is uncountable.

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6. Let $A = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$, $B = [3,5] = \{x \in \mathbb{R} : 3 \leq x \leq 5\}$, and $C = [1,6) = \{x \in \mathbb{R} : 1 \leq x < 6\}$.

   (a) Show that there exists a bijection between A and B.

   (b) Show that there exists a bijection between A and C.

   (c) Show that there exists a bijection between B and C.

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7. The goal of this problem is to give yet another proof that $\mathbb{Q}$ is countable.

   (a) Let $f : \mathbb{Q} \to \mathbb{Z}_+$ be the map defined by $f((-1)^k a/b) = 2^k 3^a 5^b$, where $a \geq 0$, $b > 0$, a/b is in lowest terms, and $k \in \{0,1\}$. Show that f is one-to-one.

   (b) Show that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Z}_+$. [Hint: Use Cantor-Schröder-Bernstein with $g : \mathbb{Z}_+ \to \mathbb{Q}$ defined by $g(n) = n$.]

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8. The goal of this problem is to show that there exists a bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$.

   (a) Show that there exists a one-to-one function $f : [0,1] \to [0,1] \times [0,1]$.

   (b) Consider the function $g : [0,1] \times [0,1] \to [0,1]$ where $g(0.d_1d_2d_3\ldots, 0.e_1e_2e_3\ldots) = 0.1d_1e_11d_2e_21d_3e_3\ldots$, where we always choose the decimal expansion ending in a string of 9s if there is a choice. Show that g is one-to-one.

   (c) Deduce that there exists a bijection between $[0,1]$ and $[0,1] \times [0,1]$.

   (d) Show that there exists a bijection between $[0,1]$ and $\mathbb{R}$. [Hint: Use $f(x) = x$ and $g(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$.]

   (e) Deduce that there exists a bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$. [Hint: Apply (d) to each copy of $[0,1]$ in (c).]

   Remark: Some other results related to these are (i) there exists a *continuous* onto function $f : [0,1] \to [0,1] \times [0,1]$ (such functions are called "space-filling curves"), but (ii) there does not exist a continuous bijection $f : [0,1] \to [0,1] \times [0,1]$. Also, (iii) there exists an additive bijection $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathbb{R}$.

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9. A real number r is <u>algebraic</u> if it is a root of a nonzero polynomial with integer coefficients, meaning that $p(r) = 0$ for some polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_n, a_{n-1}, \ldots, a_0$ are integers with $a_n \neq 0$. We say r is <u>transcendental</u> if it is not algebraic. For example, $3/4$ is algebraic because it is a root of $p(x) = 3x - 4$, and $\sqrt{2}$ is algebraic because it is a root of $p(x) = x^2 - 2$.

   (a) Let P_n be the set of nonzero polynomials of degree at most n whose coefficients are integers that are at most n in absolute value. Show that the cardinality of P_n is $(2n+1)^{n+1} - 1$.

   (b) Let S_n be the set of roots of the polynomials in the set P_n defined in part (a). Show that S_n is finite and that the set A of algebraic numbers is the union $\cup_{n \geq 1} S_n$. [Hint: Use the fact that a nonzero polynomial of degree n has at most n different roots.]

   (c) Show that the set of algebraic numbers is countable. Deduce that there are uncountably many transcendental numbers.

   Remark: The result of (c) is what is termed "non-constructive": it proves that transcendental numbers exist without actually giving any examples. In fact, it is usually quite difficult to prove that any specific real number r actually is transcendental.

   Remark: The ideas from this problem can be adapted to show that the cardinality of the "computable numbers" (real numbers that can be computed to arbitrarily good accuracy by a finite, terminating algorithm) is countable, and thus that there are uncountably many uncomputable numbers.
