E. Dummit's Math 1365 ∼ Intensive Mathematical Reasoning, Fall 2023 ∼ Homework 1, due Tue Sep 12th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

   (a) Proposition: The integer 2 is odd.
   Proof: It is a fact about odd integers that any odd integer plus any odd integer always gives an even integer. Because $2+2=4$, and 4 is an even integer, this means 2 must be odd.

   (b) Proposition: Every positive integer n, when its value is written out in English, shares at least one letter in common with the value of $n+1$ when written out in English.
   Proof: 1 and 2 share the letter o, 2 and 3 share the letter t, 3 and 4 share the letter r, 4 and 5 share the letter f, 5 and 6 share the letter i, 6 and 7 share the letter s, and all integers greater than or equal to 7 share the letter e.

   (c) Proposition: If x is an integer and $3x-2=7$, then $x=3$.
   Proof: Suppose $x=3$. Then $3x-2=3(3)-2=7$. Therefore, if $3x-2=7$, then $x=3$.

   (d) Proposition: Every odd number greater than 1, except for 9, is prime.
   Proof: Clearly, 3 is prime, 5 is prime, 7 is prime, 9 is not prime, 11 is prime, and 13 is prime. Since we have excluded 9, all odd numbers greater than 1 are prime.

   (e) Proposition: For an integer m, m is even if and only if m^2 is even.
   Proof: Suppose m is even. Then $m=2k$ for some integer k, meaning that $m^2=(2k)^2=4k^2=2\cdot 2k^2$ and thus m^2 is even. Therefore, m is even if and only if m^2 is even.

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2. Find a counterexample to the following statement: if p is a prime number, then 2^p-1 is also a prime number.

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3. After discussing Euclid's proof that there are infinitely many primes, it is sometimes claimed (occasionally in actual textbooks) that if p_1, p_2, ... , p_k is a list of the first k primes, then the number $p_1p_2\cdots p_k+1$ is always prime. Show that this statement is false by giving an explicit counterexample. (You may want to use a computer!)

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4. Write explicitly the converse, inverse, and contrapositive of the following conditional statements:

   (a) If you do not study for your exams, then you will get bad grades.

   (b) If n is an odd integer greater than 7, then n is the sum of three odd primes.

   (c) If you want to bake a cake, then you must have eggs and flour.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Suppose that P, Q, and R are any propositions.

   (a) Prove that if P and $P \Rightarrow Q$ are both true, then Q is also true. [Hint: Use a truth table to identify all cases in which both P and $P \Rightarrow Q$ are true.]

   (b) Suppose that the statements "If it is raining, then it is cloudy" and "It is raining" are both true. Is the statement "It is cloudy" necessarily true? Explain. [Hint: Use (a).]

   (c) Prove that if $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is also true.

   (d) Suppose that the statements "If it is raining, then it is cloudy" and "If it is cloudy, then people want to stay home" are both true. Is the statement "If it is raining, then people want to stay home" necessarily true? Explain. [Hint: Use (c).]

   Remarks: The results of (b) and (d) here are forms of reasoning we often apply in deductive proofs (formally, (b) is known as *modus ponens* and (d) is known as *transitivity*). The point of this problem is to verify that these forms of deduction are in fact logically correct.

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6. Using a truth table or otherwise, determine whether each of following pairs of statements are equivalent. For those that are false, give an explicit counterexample (i.e., truth values for the propositions showing the statements are different):

   (a) $A \wedge (A \vee B)$ and A.

   (b) $\neg(A \vee \neg B) \Rightarrow \neg B$ and $B \Rightarrow A$.

   (c) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ and $P \Rightarrow (P \Leftrightarrow Q)$.

   (d) $\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R)$ and True.

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7. Suppose $P_1, P_2, \ldots, P_n$ are propositions. Observe that there are 2^n rows in a truth table for $P_1, P_2, \ldots, P_n$.

   (a) For any single row R in the truth table, describe a Boolean sentence involving $P_1, P_2, \ldots, P_n$ and the connectives $\neg$ and $\wedge$ that is true in row R and false in all other rows.

   (b) For any assignment of true and false values to the 2^n rows in the truth table, show that there exists a Boolean sentence involving $P_1, \ldots, P_n$ and the connectives $\neg$, $\wedge$, and $\vee$ taking the assigned value in each row. [Hint: Join sentences from (a) with $\vee$.]

   (c) Define the connective $\uparrow$ as $P \uparrow Q = \neg(P \wedge Q)$. Show that $P \vee Q$ is logically equivalent to $(P \uparrow P) \uparrow (Q \uparrow Q)$.

   (d) With $\uparrow$ defined as in part (c), show that $\neg P$ and $P \wedge Q$ can also be expressed as statements involving only the connective $\uparrow$.

   (e) Deduce that for any assignment of true and false values to the 2^n rows in the truth table, show that there exists a Boolean sentence involving only $P_1, \ldots, P_n$ and the connective $\uparrow$ taking the assigned value in each row.
