0.1 Logic and Proof Methods

1. Let P, Q, and R be propositions.
   (a) Show that the statement $P \wedge \neg[Q \vee (R \Rightarrow P)]$ is always false.
   (b) Show that the propositions $(P \Rightarrow Q) \Leftrightarrow R$ and $P \Rightarrow (Q \Leftrightarrow R)$ are not logically equivalent.
   (c) Show that the propositions $\neg[Q \wedge \neg(P \wedge Q)] \wedge \neg P$ and $\neg Q \wedge \neg P$ are logically equivalent.

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2. Write a negation for each of the following statements:
   (a) $\forall x \forall y \exists z,\ x + y + z > 5$.
   (b) Every integer is a rational number.
   (c) $\forall x \in A\, \forall y \in B,\ x \cdot y \in A \cap B$.
   (d) There is a perfect square that is not even.
   (e) The integer n is a prime number and $n < 10$.
   (f) $\forall \epsilon > 0\, \exists \delta > 0,\ (|x - a| < \delta) \Rightarrow (|x^2 - a^2| < \epsilon)$.
   (g) For any $x \in \mathbb{R}$ there exists an $n \in \mathbb{Z}$ such that $x < n$.
   (h) There exist positive integers a and b with $2 = (a/b)^3$.

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3. Find the truth values of the following statements, where the universal set is $\mathbb{R}$:
   (a) $\forall x \forall y,\ y \neq x$.
   (b) $\forall x \exists y,\ y \neq x$.
   (c) $\exists x \forall y,\ y \neq x$.
   (d) $\exists x \exists y,\ y \neq x$.
   (e) $\forall x \forall y,\ y^2 \geq x$.
   (f) $\forall x \exists y,\ y^2 \geq x$.
   (g) $\exists x \forall y,\ y^2 \geq x$.
   (h) $\exists x \exists y,\ y^2 \geq x$.

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4. With universal set $\mathbb{Z}_+$, let $E(n)$ be the statement that n is even and let $S(n)$ be the statement that n is a perfect square greater than 1. Consider the statement
$$\forall n\ [E(n) \wedge (n > 2)] \Rightarrow [\exists m\, S(m) \wedge m | n].$$
   (a) Show that the statement is false by giving a counterexample.
   (b) Give the negation of this statement, simplified as much as possible.

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5. Write, and then prove, the contrapositive of each of these statements (assume n refers to an integer):
   (a) Suppose $a, b \in \mathbb{Z}$. If $ab = 1$ then $a \leq 1$ or $b \leq 1$.
   (b) If $5n + 1$ is even, then n is odd.
   (c) If n^3 is odd, then n is odd.
   (d) If n is not a multiple of 3, then n cannot be written as the sum of 3 consecutive integers.
   (e) Suppose $a, b \in \mathbb{Z}$. If n does not divide ab, then n does not divide a and n does not divide b.

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6. Find a counterexample to each of the following statements:
   (a) For any integers a, b, and c, if $a|b$ and $a|c$, then $b|c$.
   (b) If p and q are prime, then $p + q$ is never prime.
   (c) There do not exist integers a and b with $a^2 - b^2 = 7$.
   (d) If $n > 1$ is an integer, then $\sqrt{n}$ is always irrational.
   (e) If $n \neq 3$ then $n^2 \neq 9$.
   (f) There are no integers m, n with $m^2 - 2n^2 = 1$.
   (g) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y^4 = x$.
   (h) Two perfect squares never sum to a perfect cube.

0.2 Sets

In these problems, $\emptyset$ denotes the empty set, $\overline{A} = A^c = \{x : x \notin A\}$ denotes the complement of a set inside a universal set, and $A \backslash B = A - B = \{x \in A : x \notin B\}$ denotes set difference.

1. Let A, B, and C be sets. Prove that $(A \backslash B) \cup (B \backslash C) \subseteq (A \cup B) \backslash (B \cap C)$.

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2. Let A and B be sets. Prove that $A - B = \emptyset$ if and only if $A \subseteq B$.

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3. For any sets A, B, C, prove $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

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4. For any sets A, B inside a universal set U, prove $A \cup B^c = U$ if and only if $A^c \cap B = \emptyset$.

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5. For any sets A, B, C, prove $A \subseteq B \cup C$ if and only if $A - B \subseteq C$.

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6. Suppose A, B, and C are arbitrary sets contained in a universal set U. Identify which statements are true and which are false. Then prove the true statements and give a counterexample for the false ones.

   (a) $(A \cup B) - A = B - A$.

   (b) $A \backslash (B \cap C) = (A \backslash B) \cap (A \backslash C)$.

   (c) $\overline{A \cap B} \cup B \subseteq \overline{\overline{A} \cup B}$.

   (d) $A^c \cap B^c \subseteq (A \backslash B)^c \cap (B \backslash A)^c$.

0.3 Number Theory

1. Let m and n be positive integers.

   (a) Prove that if $m^2 + n^2$ is divisible by 4, then m and n are either both even or both odd.

   (b) Is the converse of the conditional statement in (a) true? If so prove it, and if not give a counterexample.

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2. Recall the Fibonacci numbers F_i are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$.

   (a) If F_n is the nth Fibonacci number, prove that $F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$ for every positive integer n.

   (b) Suppose $c_1 = c_2 = 2$, and for all $n \geq 3$, $c_n = c_{n-1}c_{n-2}$. Prove that $c_n = 2^{F_n}$ for every positive integer n.

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3. Suppose $a_1 = 1$ and $a_n = 3a_{n-1} + 4$ for all $n \geq 2$. Prove that $a_n = 3^n - 2$ for every positive integer n.

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4. Suppose $b_1 = 3$ and $b_{n+1} = 2b_n - n + 1$ for all $n \geq 2$. Prove that $b_n = 2^n + n$ for every positive integer n.

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5. A sequence is defined by the recurrence relation $c_n = 4c_{n-1} - 4c_{n-2}$ for $n \geq 2$, where $c_0 = 6$ and $c_1 = 8$. Prove that $c_n = (6 - 2n)2^n$ for all integers $n \geq 0$.

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6. Suppose $d_1 = 2$, $d_2 = 4$, and for all $n \geq 3$, $d_n = d_{n-1} + 2d_{n-2}$. Prove that $d_n = 2^n$ for every positive integer n.

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7. Show that $25^n + 7$ is a multiple of 8 for every positive integer n.

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8. Prove that $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=2-\frac{1}{2^n}$ for every positive integer n.

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9. Prove that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$ for every positive integer n.

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10. Calculate the greatest common divisor and least common multiple of each pair of integers:

(a) 256 and 520. (b) 921 and 177. (c) 2019 and 5678. (d) $2^3 3^2 5^4 7$ and $2^4 3^3 5^4 11$.

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11. Decide whether each residue class has a multiplicative inverse modulo m. If so, find it, and if not, explain why not:

(a) $\overline{10}$ mod 25. (b) $\overline{11}$ mod 25. (c) $\overline{12}$ mod 25. (d) $\overline{30}$ mod 42. (e) $\overline{31}$ mod 42. (f) $\overline{32}$ mod 42.

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12. Prove that the sum of any six consecutive integers is congruent to 3 modulo 6.

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13. Prove that 7^n+5 is divisible by 6 for all positive integers n.

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14. Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Show that $a(b+c) \equiv b(a+d) \pmod{n}$.

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15. Suppose n is an integer. Prove that $2|n$ and $3|n$ if and only if $6|n$.

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16. If $A=\{4a+6b : a,b\in\mathbb{Z}\}$ and $B=\{2c : c\in\mathbb{Z}\}$, prove that $A=B$.

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17. If p is a prime, prove that $\gcd(n,n+p)>1$ if and only if $p|n$.

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18. If $C=\{6c : c\in\mathbb{Z}\}$ and $D=\{10a+14b : a,b\in\mathbb{Z}\}$, prove that $C\subseteq D$.

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19. Prove that if a and b are both odd, then a^2+b^2-2 is divisible by 8.

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20. Prove that the product of two consecutive even integers is always 1 less than a perfect square.

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21. If n is any positive integer, prove that $n-1$ is invertible modulo n and its multiplicative inverse is itself.

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22. Use the Euclidean algorithm to find the multiplicative inverse of $\overline{26}$ in the multiplicative group $(\mathbb{Z}/59\mathbb{Z})^\times$ of nonzero integers modulo 59.

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23. Suppose g and h are elements of a group such that $g^{-1}h^{-1}=h^{-1}g^{-1}$. Prove that $gh=hg$.

0.4 Relations and Equivalence Relations

1. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) irreflexive, (vi) an equivalence relation, (vii) a partial ordering, and (viii) a total ordering.

   (a) $R = \{(1,1), (2,1), (2,2)\}$ on the set $\{1,2\}$.

   (b) $R = \{(1,2), (2,1)\}$ on the set $\{1,2\}$.

   (c) $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on the set $\{1,2,3,4\}$.

   (d) The divisibility relation on the set $\{2,-3,4,-5,6\}$.

   (e) The divisibility relation on the set $\{2,-4,-12,36\}$.

   (f) The relation R on $\mathbb{Z}$ with $a\ R\ b$ precisely when $|a| \equiv |b|$ modulo 5.

   (g) The relation R on $\mathbb{R}$ with $a\ R\ b$ precisely when $ab > 0$.

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2. If $R, S : A \to B$ are relations, prove that $R^{-1} \cap S^{-1} = (R \cap S)^{-1}$.

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3. Identify the ordered pairs in the equivalence relation that corresponds to the partition $\{1,2,4\}, \{3,5\}, \{6\}$ of $\{1,2,3,4,5,6\}$.

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4. Show $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : |x| = |y|\}$ is an equivalence relation on $\mathbb{Z}$ and list the equivalence classes of 0, 2, -2, 4.

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5. A relation R on integers is defined via $x\,R\,y$ when $5|(6x - y)$.

   (a) Prove that R is an equivalence relation.

   (b) Describe (as explicitly as possible) the equivalence classes into which $\mathbb{Z}$ is partitioned by R.

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6. Show that the only equivalence relation R on A that is a function from A to A is the identity relation.

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7. Suppose R is a reflexive and transitive relation on a set A. Show that $S = R \cap R^{-1}$ is an equivalence relation on A.

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8. Suppose G is a group and H is a subgroup, and define the relation R on G by saying $g_1\,R\,g_2$ whenever there exists $h \in H$ such that $g_1 = hg_2$. Prove that R is an equivalence relation.

0.5 Functions

1. For each of the following functions $f : A \to B$, identify whether (i) f is one-to-one, (ii) f is onto, (iii) f is a bijection.

   (a) $f = \{(1,2), (2,3), (3,4), (4,1)\}$ from $\{1,2,3,4\}$ to itself.

   (b) $f = \{(1,3), (2,4), (3,1), (4,4)\}$ from $\{1,2,3,4\}$ to itself.

   (c) $f(x) = 2x$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.

   (d) $f(n) = 2n$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.

   (e) $f(x) = \dfrac{x}{x-1}$ from $A = \mathbb{R}\backslash\{1\}$ to $B = \mathbb{R}$.

   (f) $f(x) = x^3$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.

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2. Let $S = \mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$ be the additive group of integers modulo 6.

   (a) Define the function $g : S \to S$ via $g(n) = 2n + 3$. Find the image of g. Is g one-to-one? Onto?
   (b) Define the function $h : S \to S$ via $g(n) = 5n - 3$. Prove that h is a bijection and find its inverse function h^{-1}.

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3. Suppose $f : A \to B$ is a function.

   (a) If f is one-to-one, show that there is a bijection between A and $\mathrm{im}(f)$. Deduce that $|A| = |\mathrm{im}(f)|$.
   (b) If A and B are both finite and $|A| = |B|$, show that f is one-to-one implies that f is onto.

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4. Define the function $f : \mathbb{Q}\backslash\{\frac{7}{2}\} \to \mathbb{Q}$ via $f(x) = \dfrac{6x + 5}{2x - 7}$.

   (a) Prove that f is one-to-one.
   (b) Find a formula for the inverse function f^{-1}.
   (c) Verify explicitly that $(f^{-1} \circ f)(x) = x$ for all x in the domain of f.
   (d) Determine the image of f. Is f onto?

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5. Let $F(x, y) = (5x + 4, y - 5)$.

   (a) Prove that $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is a bijection.
   (b) Prove that $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is not onto by finding an element of $\mathbb{Z} \times \mathbb{Z}$ missing from its image.

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6. Suppose that $f : A \to A$ is a function. Show that $f(f(a)) = a$ for all $a \in A$ if and only if f^{-1} exists and $f^{-1}(a) = f(a)$ for all $a \in A$.

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7. Suppose $f : B \to C$ is one-to-one. If $g, h : A \to B$ have $f \circ g = f \circ h$, show that $g = h$.

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8. Suppose that $f : B \to C$ and $g : A \to B$ are functions.

   (a) If f and g are one-to-one, prove that $f \circ g$ is also one-to-one.
   (b) If f and g are onto, prove that $f \circ g$ is also onto.

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9. Let S and T be any sets and let $f : S \to T$ be a function. Recall that for a subset A of S, we define $f(A) = \{f(a) : a \in A\}$ and for a subset C of T, we define $f^{-1}(C) = \{a \in A : f(a) \in C\}$.

   (a) For any subset A of S, show that $A \subseteq f^{-1}(f(A))$.
   (b) If $f : A \to B$ is one-to-one and A is a subset of S, prove that $A = f^{-1}(f(A))$.
   (c) For any subset C of T, show that $f(f^{-1}(C)) \subseteq C$.
   (d) If f is onto and C is a subset of T, prove that $f(f^{-1}(C)) = C$.
   (e) If f is one-to-one and A and B are subsets of S, prove that $f(A) \cap f(B) \subseteq f(A \cap B)$.

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10. Suppose $f : A \to B$ is a bijection. Show that $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ given by $\tilde{f}(S) = \{f(s) : s \in S\}$ is also a bijection, where $\mathcal{P}(S)$ denotes the power set of S (the set of subsets of S).

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11. Let S be the set of equivalence classes of an equivalence relation R on A and define the function $f : A \to S$ via $f(a) = [a]$. Show that f is one-to-one if and only if R is the identity relation.

0.6 Cardinality and Counting

1. How many integers less than or equal to 251 are divisible by 4 or by 5 or by 7?

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2. Suppose that S and T are sets such that $|S| = 8$ and $|T| = 11$.

   (a) How many relations are there from S to T?
   (b) How many functions are there from S to T?
   (c) How many functions are there from T to S?
   (d) How many one-to-one functions are there from S to T?
   (e) How many one-to-one functions are there from T to S?

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3. A set S consists of 73 positive integers. What is the minimum number of elements of S that belong to the same remainder class upon division by 8?

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4. Suppose A and B are sets with $|A| = 2$ and $|B| = 8$.

   (a) How many onto functions are there from A to B?
   (b) How many onto functions are there from B to A?

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5. Prove that if A is countable and B is uncountable, then the set difference $B - A = B \backslash A$ is uncountable.

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6. Prove that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$, the set of rational numbers strictly between 0 and 1.

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7. Prove that the set $\mathbb{Q} \times \mathbb{Z}$ is countable and that the set $\mathbb{R} \times \mathbb{Z}$ is uncountable.

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8. Prove that the set of all finite subsets of $\mathbb{Z}$ is countable.

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9. Use the Cantor-Schröder-Bernstein theorem to prove that there exists a bijection between the half-closed interval $[1, 7) = \{x \in \mathbb{R} : 1 \leq x < 7\}$ and the open interval $(2, 9) = \{x \in \mathbb{R} : 2 < x < 9\}$.

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10. Prove that there exists a bijection between $(0, 1)$ and $[0, 1]$. [Hint: Cantor-Schröder-Bernstein.]
