Note: These outline answers do not contain full details of all solutions, and would in some cases lack enough detail and justification to receive full credit as actual exam responses. The intention is to use these outline responses as a guide for the major pieces in the solution to each problem: if you have previously made an effort to solve a problem, the outline answer should provide enough information to help you complete your solution.

0.1 Logic and Proof Methods

1. Each of these can be solved by drawing a truth table. We give alternative approaches.
   (a) $P \wedge \neg[Q \vee (R \Rightarrow P)] = P \wedge \neg[Q \vee \neg R \vee P] = P \wedge \neg Q \wedge R \wedge \neg P$ which is false due to the $P \wedge \neg P$.
   (b) When P is true, Q is false, R is true, then $(P \Rightarrow Q) \Leftrightarrow R$ is false while $P \Rightarrow (Q \Leftrightarrow R)$ is true.
   (c) $\neg[Q \wedge \neg(P \wedge Q)] \wedge \neg P = [\neg Q \vee (P \wedge Q)] \wedge \neg P = (\neg Q \wedge \neg P) \vee (P \wedge Q \wedge \neg P) = (\neg Q \wedge \neg P) \vee \text{False} = \neg Q \wedge \neg P$.

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2. (a) $\exists x \exists y \forall z$, $x + y + z \leq 5$
   (b) There exists an integer that is not rational.
   (c) $\exists x \in A \, \exists y \in B$, $x \cdot y \notin A \cap B$.
   (d) Every perfect square is even.
   (e) The integer n is either not prime or $n \geq 10$.
   (f) $\exists \epsilon > 0 \, \forall \delta > 0$, $(|x - a| < \delta) \wedge (|x^2 - a^2| \geq \epsilon)$.
   (g) There exists an $x \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$, $x \geq n$.
   (h) For all positive integers a and b, $2 \neq (a/b)^3$.

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3. (a) False (b) True (c) False (d) True (e) False (f) True (g) True (h) True

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4. (a) In words, any even integer $n > 2$ has some perfect square $m > 1$ that divides n. So $n = 6$ is a counterexample since there is no square other than 1 that divides 6.
   (b) $\exists n \; E(n) \wedge (n > 2) \wedge [\forall m \, S(m) \Rightarrow m \nmid n]$: there exists an even integer $n > 2$ such that for all squares $m > 1$, m does not divide n.

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5. (a) If $a > 1$ and $b > 1$, then $ab \neq 1$. Proof: If $a > 1$ and $b > 1$ then multiplying $a > b$ by b yields $ab > b > 1$ so $ab > 1$. In particular $ab \neq 1$.
   (b) If n is even, then $5n + 1$ is odd. Proof: If $n = 2k$ then $5n + 1 = 10k + 1 = 2(5k) + 1$ is odd by definition.
   (c) If n is even then n^3 is even. Proof: If $n = 2k$ then $n^3 = 8k^3 = 2(4k^3)$ is even by definition.
   (d) If n is the sum of 3 consecutive integers, then n is a multiple of 3. Proof: If $n = a + (a + 1) + (a + 2)$ then $n = 3a + 3 = 3(a + 1)$ is a multiple of 3.
   (e) If n divides a or n divides b then n divides ab. Proof: If $n|a$ then $a = kn$ so $ab = (kb)n$, and if $n|b$ then $b = ln$ so $ab = (al)n$. In either case, $n|ab$.

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6. There are many examples for each part. Here is one for each:
   (a) Example: $a = 2$, $b = 4$, $c = 6$.
   (b) Example: $p = 2$, $q = 3$, then $p + q = 5$ is prime.
   (c) Example: $a = 4$, $b = 3$, then $a^2 - b^2 = 16 - 9 = 7$.
   (d) Example: $\sqrt{4} = 2$ is rational.
   (e) Example: $n = -3$, then $n \neq 3$ but $n^2 = 9$.
   (f) Example: $m = 3$, $n = 2$, then $m^2 - 2n^2 = 9 - 8 = 1$.
   (g) Example: $x = -1$, then there is no possible y with $y^4 = x$.
   (h) Examples: $2^2 + 2^2 = 2^3$, or $5^2 + 10^2 = 5^3$.

0.2 Sets

1. Let $x \in (A\backslash B) \cup (B\backslash C)$. Then $x \in A\backslash B$ or $x \in B\backslash C$. If $x \in A\backslash B$ then $x \in A$ and $x \notin B$ so $x \in A \cup B$ and $x \notin B \cap C$, meaning $x \in (A \cup B)\backslash(B \cap C)$. If $x \in B\backslash C$ then $x \in B$ and $x \notin C$ so $x \in A \cup B$ and $x \notin B \cap C$, so again $x \in (A \cup B)\backslash(B \cap C)$.

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2. First suppose $A - B = \emptyset$. If $x \in A$ then since $A - B$ is empty, x must be in B (otherwise x would be in $A - B$), so $A \subseteq B$. Conversely, if $A \subseteq B$, then there are no elements of A not in B, so $A - B = \emptyset$.

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3. Note $x \in A\backslash(B \cap C) \iff x \in A$ and $x \notin (B \cap C) \iff x \in A$ and ($x \notin B$ or $x \notin C$) $\iff$ ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) $\iff x \in A\backslash B$ or $x \in A\backslash C \iff x \in (A\backslash B) \cup (A\backslash C)$.

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4. Observe $(A \cup B^c)^c = A^c \cap (B^c)^c = A^c \cap B$ by de Morgan's laws, so $A \cup B^c$ and $A^c \cap B$ are complements. Thus, if $A \cup B^c = U$ then $A^c \cap B = U^c = \emptyset$ and conversely if $A^c \cap B = \emptyset$ then $A \cup B^c = \emptyset^c = U$.

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5. First suppose $A \subseteq B \cup C$. If $x \in A - B$ then $x \in A$ and $x \notin B$. Since $A \subseteq B \cup C$, $x \in B \cup C$ so $x \in B$ or $x \in C$ but since $x \notin B$ we must have $x \in C$: thus $A - B \subseteq C$. Conversely suppose $A - B \subseteq C$ and let $x \in A$. If $x \in B$ then clearly $x \in B \cup C$ and otherwise if $x \notin B$ then $x \in A - B$ hence $x \in C$ and once again $x \in B \cup C$: thus $A \subseteq B \cup C$.

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6. (a) True. Note $x \in (A \cup B) - A$ iff $x \in (A \cup B) \cap A^c$ iff $x \in B \cap A^c$ iff $x \in B - A$.
   (b) False. Counterexample: $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$. Then $A\backslash(B \cap C) = \{1, 2\}$ while $(A\backslash B) \cap (A\backslash C) = \emptyset$.
   (c) False. Counterexample: $A = \{1\}$, $B = \{1, 2\}$ with $U = \{1, 2\}$. Then $\overline{A \cap B} \cup B = \{1, 2\}$ while $\overline{\overline{A} \cup B} = \{1\}$.
   (d) True. Note $(A\backslash B)^c = (A \cap B^c)^c = A^c \cup B$, and similarly $(B\backslash A)^c = A \cup B^c$. If $x \in A^c \cap B^c$ then $x \in A^c \cup B$ and also $x \in A \cup B^c$.

0.3 Number Theory

1. (a) Since a, a^2 have the same parity, $m^2 + n^2$ has the same parity as $m + n$. So if $m^2 + n^2$ is even then $m + n$ must be even, meaning m, n have the same parity.
   (b) It is false: if $m = n = 1$ then $m^2 + n^2 = 2$ is not divisible by 4.

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2. (a) Induct on n. Base case $n = 1$ has $F_1 + F_3 = 3 = F_4$. Inductive step: if $F_1 + \cdots + F_{2n+1} = F_{2n+2}$ then $F_1 + \cdots + F_{2n+1} + F_{2n+3} = [F_1 + \cdots + F_{2n+1}] + F_{2n+3} = F_{2n+2} + F_{2n+3} = F_{2n+4}$ as required.
   (b) Induct on n. Base cases $n = 1$ and $n = 2$ have $c_1 = 2^{F_1}$ and $c_2 = 2^{F_2}$. Inductive step: if $c_n = 2^{F_n}$ and $c_{n-1} = 2^{F_{n-1}}$ then $c_{n+1} = c_n c_{n-1} = 2^{F_n} 2^{F_{n-1}} = 2^{F_n + F_{n-1}} = 2^{F_{n+1}}$ as required.

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3. Induct on n. Base case $n = 1$ has $a_1 = 3^1 - 2$. Inductive step: if $a_n = 3^n - 2$ then $a_{n+1} = 3(3^n - 2) + 4 = 3^{n+1} - 2$.

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4. Induct on n. Base case $n = 1$ has $b_1 = 2^1 + 1$. Inductive step: if $b_n = 2^n + n$ then $b_{n+1} = 2(2^n + n) - n + 1 = 2^{n+1} + (n + 1)$.

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5. Induct on n. Base cases $n = 0$ and $n = 1$ have $c_0 = 6 \cdot 2^0$ and $c_1 = 4 \cdot 2^1$. Inductive step: if $c_n = (6 - 2n)2^n$ and $c_n = (6 - 2(n-1))2^{n-1} = (4 - n)2^n$ then $c_{n+1} = 4(6 - 2n)2^n - 4(4 - n)2^n = (24 - 8n - 16 + 4n)2^n = (8 - 4n)2^n = (6 - 2(n + 1))2^{n+1}$ as required.

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6. Induct on n . Base cases $n = 1$ and $n = 2$ have $d_1 = 2^1$ and $d_2 = 2^2$. Inductive step: if $d_n = 2^n$ and $d_{n-1} = 2^{n-1}$ then $d_{n+1} = 2^n + 2(2^{n-1}) = 2^n + 2^n = 2^{n+1}$ as required.

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7. Induct on n. Base case $n = 1$ has $25^1 + 7 = 32$ a multiple of 8. Inductive step: if 8 divides $25^n + 7$, then 8 divides $25 \cdot (25^n + 7) - 24 \cdot 7 = 25^{n+1} + 7$. (Reducing modulo 8 also works.)

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8. Induct on n. Base case $n = 1$ has $1 = 2 - 1/2^0$. Inductive step: If $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$, then $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}$ as required.

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9. Induct on n. Base case $n = 1$ has $\frac{1}{1 \cdot 2} = \frac{1}{2}$. Inductive step: if $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$ then $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ as required.

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10. In (a)-(c), the Euclidean algorithm allows for a quick calculation of the gcd, and then $\text{lcm}(a, b) = ab/\gcd(a, b)$.

    (a) $520 = 2 \cdot 256 + 8$, $256 = 32 \cdot 8$, so gcd is 8, and lcm is $256 \cdot 520/8$.

    (b) $921 = 5 \cdot 177 + 36$, $177 = 4 \cdot 36 + 33$, $36 = 1 \cdot 33 + 3$, $33 = 11 \cdot 3$ so gcd is 3, and lcm is $921 \cdot 177/3$.

    (c) $5678 = 2 \cdot 2019 + 1640$, $2019 = 1 \cdot 1640 + 379$, $1640 = 4 \cdot 379 + 124$, $379 = 3 \cdot 124 + 7$, $124 = 17 \cdot 7 + 5$, $7 = 1 \cdot 5 + 2$, $5 = 2 \cdot 2 + 1$, $2 = 2 \cdot 1$, so gcd is 1 and lcm is $2019 \cdot 5678$.

    (d) Taking the smallest exponents of each prime yields gcd $2^3 3^2 5^4$, and the largest exponents yield lcm $2^4 3^3 5^4 7 \cdot 11$.

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11. (a) Not invertible, $\gcd(10, 25) = 5 > 1$.

    (b) Invertible, $\gcd(11, 25) = 1$. By Euclid $-9 \cdot 11 + 4 \cdot 25 = 1$ so $-9 \cdot 11 \equiv 1 \pmod{25}$ so $11^{-1} \equiv -9$.

    (c) Invertible, $\gcd(12, 25) = 1$. By Euclid $-2 \cdot 12 + 1 \cdot 25 = 1$ so $-2 \cdot 12 \equiv 1 \pmod{25}$ so $12^{-1} \equiv -2$.

    (d) Not invertible, $\gcd(30, 42) = 6 > 1$.

    (e) Invertible, $\gcd(31, 42) = 1$. By Euclid $19 \cdot 31 - 14 \cdot 42 = 1$ so $19 \cdot 31 \equiv 1 \pmod{42}$ so $31^{-1} \equiv 19$.

    (f) Not invertible, $\gcd(32, 42) = 2 > 1$.

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12. If n is the sum of $k, k+1, k+2, k+3, k+4, k+5$ then $n = 6k + 15 \equiv 3 \pmod 6$. Conversely if $n \equiv 3 \bmod 6$ so that $n = 3 + 6a$, then n is the sum of $a - 2, a - 1, a, a + 1, a + 2, a + 3$.

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13. Modulo 6 we have $7^n + 5 \equiv 1^n + 5 \equiv 1 + 5 \equiv 0 \pmod 6$, which means $7^n + 5$ is divisible by 6.

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14. Since $a \equiv b \pmod n$ and $c \equiv d \pmod n$ we see $b + c \equiv a + d \pmod n$. Then $a(b + c) \equiv b(b + c) \equiv b(a + d) \pmod n$ so $a(b + c) \equiv b(a + d) \pmod n$.

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15. Clearly, if $6|n$ then $2|n$ and $3|n$. For the other direction, if $2|n$ then $n = 2k$. Then if $3|2k$ we must have $3|k$ since $3 \nmid 2$ and 3 is prime. So $k = 3a$, and thus $n = 6a$, meaning $6|n$.

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16. First, $A \subseteq B$ because if $n = 4a + 6b$ then $n = 2(2a + 3c) \in B$. Also, $B \subseteq A$ because if $n = 2c$ then we would have $n = 4(2c) + 6(-c) \in A$ via Euclidean algorithm calculation.

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17. Note $\gcd(n, n + p) = \gcd(n, p)$ by gcd properties. Then $\gcd(n, p)$ divides p so is either 1 or p, and it is equal to p if and only if $p|n$ (by definition of gcd).

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18. If $n \in C$, then $n = 6c$ for some c. Then $n = 10(2c) + 14(-c) \in D$ as required.

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19. If $a = 2c + 1$ and $b = 2d + 1$ then $a^2 + b^2 - 2 = 4(c^2 + c + d^2 + d)$, which is divisible by 8 since $c^2 + c = c(c + 1)$ is always even as is $d^2 + d$.

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20. Note $(2n)(2n + 2) = 4n^2 + 4n$ is 1 less than $(2n + 1)^2 = 4n^2 + 4n + 1$.

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21. Note $n - 1 \equiv -1 \pmod n$ so $(n - 1)^{-1} \equiv (-1)^{-1} \equiv -1 \equiv n - 1 \pmod n$. Or, $(n - 1)^2 = n^2 - 2n + 1 \equiv 1 \pmod n$.

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22. We have $59 = 2 \cdot 26 + 7$, $26 = 3 \cdot 7 + 5$, $7 = 1 \cdot 5 + 2$, $5 = 2 \cdot 2 + 1$, $2 = 2 \cdot 1$. Solving for remainders gives $7 = 59 - 2 \cdot 26$, $5 = -3 \cdot 59 + 7 \cdot 26$, $2 = 4 \cdot 59 - 9 \cdot 26$, $1 = -11 \cdot 59 + 25 \cdot 26$. So $25 \cdot 26 \equiv 1 \pmod{59}$ so $\overline{26}^{-1} = \overline{25}$.

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23. Note $(g^{-1}h^{-1})^{-1} = (h^{-1})^{-1}(g^{-1})^{-1} = hg$ and $(h^{-1}g^{-1})^{-1} = (g^{-1})^{-1}(h^{-1})^{-1} = gh$. So taking the inverse of $g^{-1}h^{-1} = h^{-1}g^{-1}$ yields $gh = hg$.

0.4 Relations and Equivalence Relations

1.

#	Reflexive	Symmetric	Transitive	Antisymmetric	Irreflexive	Equiv Rel	Partial	Total
(a)	Yes	No	Yes	Yes	No	No	Yes	Yes
(b)	No	Yes	No	No	Yes	No	No	No
(c)	Yes	Yes	Yes	No	No	Yes	No	No
(d)	Yes	No	Yes	Yes	No	No	Yes	No
(e)	Yes	No	Yes	Yes	No	No	Yes	Yes
(f)	Yes	Yes	Yes	No	No	Yes	No	No
(g)	No (0)	Yes	Yes	No	No	No	No	No

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2. Note $(a,b) \in R^{-1} \cap S^{-1} \iff (a,b) \in R^{-1}$ and $(a,b) \in S^{-1} \iff (b,a) \in R$ and $(b,a) \in S \iff (b,a) \in R \cap S \iff (a,b) \in (R \cap S)^{-1}$.

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3. $R = [\{1,2,4\} \times \{1,2,4\}] \cup [\{3,5\} \times \{3,5\}] \cup [\{6\} \times \{6\}]$
$= \{(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(4,1),(4,2),(4,4),(3,3),(3,5),(5,3),(5,5),(6,6)\}$.

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4. R is reflexive since $|x| = |x|$, R is symmetric since $|x| = |y|$ implies $|y| = |x|$, and R is transitive since $|x| = |y|$ and $|y| = |z|$ imply $|x| = |z|$. Also, $[0] = \{0\}$, $[2] = [-2] = \{2,-2\}$, $[4] = \{4,-4\}$.

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5. (a) This relation says $x\,R\,y$ when $6x \equiv y \pmod 5$, or equivalently when $x \equiv y \pmod 5$. So this relation is just congruence modulo 5, which we already know is an equivalence relation.

   (b) The equivalence classes are simply the congruence classes modulo 5: $[n] = \{\ldots, n-10, n-5, n, n+5, n+10, \ldots\} = \{n+5k : k \in \mathbb{Z}\}$.

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6. If R is reflexive and a function, then $R(a) = a$ for all $a \in A$, so the only possibility is to have $R(a) = a$ for all $a \in A$. But clearly the identity function is also an equivalence relation, so it is the only one that works.

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7. Reflexive: For each $a \in A$ we have $(a,a) \in R$ and so $(a,a) \in R^{-1}$ hence $(a,a) \in S$. Symmetric: if $(a,b) \in S$ then $(a,b) \in R$ and $(a,b) \in R^{-1}$ so $(b,a) \in R^{-1}$ and $(b,a) \in R$ so $(b,a) \in S$. Transitive: if $(a,b),(b,c) \in S$ then $(a,b),(b,c) \in R$ so $(a,c) \in R$ and also $(c,b),(b,a) \in R$ so $(c,a) \in R$ so $(a,c) \in R^{-1}$ so $(a,c) \in S$.

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8. Reflextive: $e \in H$ and $g_1 = eg_1$ so $g_1\,R\,g_1$. Symmetric: If $g_1\,R\,g_2$ so that $g_1 = hg_2$ with $h \in H$ then $h^{-1}g_1 = g_2$ and $h^{-1} \in H$, so $g_2\,R\,g_1$. Transitive: If $g_1\,R\,g_2$ and $g_2\,R\,g_3$ so that $g_1 = hg_2$ and $g_2 = kg_3$ with $h,k \in H$ then $g_1 = hg_2 = hkg_3$ and $hk \in H$ so $g_1\,R\,g_3$.

0.5 Functions

1. (a) f is one-to-one, onto, and a bijection since its inverse is also a function.

   (b) f is not one-to-one since $f(2) = f(4)$ and f is not onto since $\text{im}(f)$ misses 2.

   (c) f is one-to-one, onto, and a bijection since it has an inverse $f^{-1}(x) = x/2$.

   (d) f is one-to-one but not onto since $\text{im}(f)$ is only the even integers.

   (e) f is one-to-one but not onto since its image misses 1.

   (f) f is one-to-one, onto, and a bijection since it has an inverse $f^{-1}(x) = x^{1/3}$.

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2. (a) Note $g(0) = g(3) = 3$, $g(1) = g(4) = 5$, $g(2) = g(5) = 1$ so $\text{im}(g) = \{1, 3, 5\}$. Then g is not one-to-one since $g(0) = g(3)$ and g is not onto since $0, 2, 4$ are not in $\text{im}(g)$.
   (b) We have $h^{-1}(n) = 5^{-1}(n + 3) = 5(n + 3) = 5n + 3$ since $5^{-1} \equiv 5$ modulo 6. Since h^{-1} exists, h is a bijection.

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3. (a) The function $g : A \to \text{im}(f)$ with $g(a) = f(a)$ for all $a \in A$ is one-to-one and onto hence a bijection. Then $|A| = |\text{im}(f)|$ by the definition of cardinality.
   (b) If f is one-to-one then by (a), $|A| = |\text{im}(f)|$. Since $|A| = |B|$ and A and B are finite, this means $\text{im}(f)$ is a subset of the finite set B having the same cardinality as B: thus $\text{im}(f) = B$ so f is onto.

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4. (a) If $f(a) = f(b)$ then $\dfrac{6a+5}{2a-7} = \dfrac{6b+5}{2b-7}$ so $(6a + 5)(2b - 7) = (2a - 7)(6b + 5)$ so $12ab + 10b - 42a - 35 = 12ab - 42b + 10a - 45$ so $52a = 52b$ so $a = b$.
   (b) Solving $y = \dfrac{6x+5}{2x-7}$ for x yields $y(2x - 7) = 6x + 5$ so $2xy - 7y = 6x + 5$ so $x = \dfrac{7y+5}{2y-6}$. So $f^{-1}(y) = \dfrac{7y+5}{2y-6}$.
   (c) $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(\dfrac{6x+5}{2x-7}) = \dfrac{7\frac{6x+5}{2x-7}+5}{2\frac{6x+5}{2x-7}-7} = \dfrac{7(6x+5)+5(2x-7)}{2(6x+5)-7(2x-7)} = \dfrac{52x}{52} = x$ as claimed.
   (d) The image of f is the domain of f^{-1}, which is $\mathbb{Q}\backslash\{3\}$ from (b). So f is not onto as the image omits 3.

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5. (a) F is a bijection as it has an inverse function $G(x, y) = (\frac{x-4}{5}, y + 5)$: note $F(G(x, y)) = F(\frac{x-4}{5}, y + 5) = (x, y)$ and $G(F(x, y)) = G(5x + 4, y - 5) = (x, y)$.
   (b) There is no $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ with $F(a, b) = (0, 0)$ since this would require $a = -4/5$ which is not an integer.

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6. Note $f(f(a)) = a$ for all $a \in A \iff f \circ f = i_A \iff f^{-1} = f$ as functions on $A \iff f^{-1}$ exists and $f^{-1}(a) = f(a)$ for all $a \in A$.

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7. Let $x \in A$. Then by hypothesis $(f \circ g)(x) = (f \circ h)(x)$ which means $f(g(x)) = f(h(x))$. But f is one-to-one, so this implies $g(x) = h(x)$. Since g and h agree on all elements in A, that means $g = h$.

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8. (a) Suppose $(f \circ g)(a_1) = (f \circ g)(a_2)$ for some $a_1, a_2 \in A$, so that $f(g(a_1)) = f(g(a_2))$. Since f is one-to-one, $f(g(a_1)) = f(g(a_2))$ implies $g(a_1) = g(a_2)$, and then since g is one-to-one, we have $a_1 = a_2$.
   (b) Let $c \in C$ be arbitrary. Since f is onto, there exists $b \in B$ such that $f(b) = c$. Then since g is onto, there exists $a \in A$ such that $g(a) = b$. Then $f(g(a)) = f(b) = c$, so $f \circ g$ is onto.

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9. (a) Suppose $a \in A$. Then $f(a) \in f(A)$, so by definition we have $a \in f^{-1}(f(A))$.
   (b) From (a), $A \subseteq f^{-1}(f(A))$. For the reverse, suppose $a \in f^{-1}(f(A))$, so that $f(a) \in f(A)$. Since f is one-to-one, $f(a) = f(b)$ implies $a = b$, so $f(a) \in f(A)$ implies $a \in A$.
   (c) Suppose $a \in f^{-1}(C)$. Then $f(a) \in C$ by definition. This holds for all $a \in f^{-1}(C)$, so $f(f^{-1}(C)) \subseteq C$.
   (d) From (c), $f(f^{-1}(C)) \subseteq C$. For the reverse, suppose $c \in C$. Since f is onto, there exists $a \in A$ with $f(a) = c$, so $a \in f^{-1}(C)$. Hence $c \in f(f^{-1}(C))$.
   (e) Suppose $x \in f(A) \cap f(B)$, meaning that $x = f(a) = f(b)$ for some $a \in A$ and $b \in B$. But since f is one-to-one this means $a = b$, and so this element a is in both A and B: thus $x = f(a)$ for some $a \in A \cap B$ so $x \in f(A \cap B)$.

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10. Note f has an inverse g. Then in fact $\tilde{f}$ has an inverse $\tilde{g} : \mathcal{P}(B) \to \mathcal{P}(A)$ with $\tilde{g}(T) = \{g(t) : t \in T\}$. Explicitly, for $S \subseteq A$, $\tilde{g}(\tilde{f}(S)) = \tilde{g}(\{f(s) : s \in S\} = \{g(f(s)) : s \in S\} = \{s : s \in S\} = S$ and $\tilde{f}(\tilde{g}(T)) = \tilde{f}(\{g(t) : t \in T\}) = \{f(g(t)) : t \in T\} = \{t : t \in T\} = T$.

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11. All equivalence relations contain the identity relation. So f is one-to-one $\iff$ $[a] = [b]$ is equivalent to $a = b \iff$ $a\,R\,b$ is equivalent to $a = b \iff$ R equals the identity relation.

0.6 Cardinality and Counting

1. Inside $U = \{1, 2, \ldots, 251\}$, if A is the set of multiples of 4, B is the set of multiples of 5, and C is the set of multiples of 7, then by inclusion-exclusion, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Since $A \cap B$ is the multiples of 20, $A \cap C$ is the multiples of 28, $B \cap C$ is the multiples of 35, and $A \cap B \cap C$ is the multiples of 140, and the number of multiples of n in U is $251/n$ rounded down to the nearest integer, the total is $62 + 50 + 35 - 12 - 8 - 7 + 1 = 121$.

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2. (a) Each of the 88 elements of $S \times T$ can be included or not, so there are 2^{88} total.
   (b) Each of the 8 elements of S has 11 possible images in T, so there are 11^8 total.
   (c) Each of the 11 elements of T has 8 possible images in S, so there are 8^{11} total.
   (d) There are 11 choices for the first value, 10 for the second, ... , and 4 for the eighth, giving $11 \cdot 10 \cdot \cdots \cdot 4 = 11!/3!$.
   (e) A function from a set of size 11 to a set of size 8 cannot be one-to-one (pigeonhole), so there are 0.

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3. If there are at most n elements in each residue class modulo 8, then there are at most $8n$ total elements. So we need $8n \geq 73$ yielding $n \geq 9.125$, so since n is an integer, that means some residue class must have at least 10 elements. Since 10 is clearly achievable (e.g., with the integers 1, 2, ... , 73) the minimum is 10.

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4. (a) There are no onto functions from A to B since the cardinality of B is larger than that of A.
   (b) If $A = \{x, y\}$ there are 2^8 functions from B to A. One has image $\{x\}$ and one has image $\{y\}$ and the other $2^8 - 2$ have image $\{x, y\}$ hence are onto.

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5. Note that B is a subset of $A \cup (B \backslash A)$. If A and $B \backslash A$ are countable then their union is also countable, hence any subset is countable. If B is uncountable then this is a contradiction, so $B \backslash A$ is uncountable.

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6. Both $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$ are countably infinite, so there is a bijection between these sets since they are both in bijection with the positive integers.

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7. The Cartesian product of two countable sets is countable, so $\mathbb{Q} \times \mathbb{Z}$ is countable since both $\mathbb{Q}$ and $\mathbb{Z}$ are countable. But $\mathbb{R} \times \mathbb{Z}$ contains $\mathbb{R} \times \{1\}$ which is in bijection with $\mathbb{R}$, so $\mathbb{R} \times \mathbb{Z}$ has an uncountable subset hence is uncountable itself.

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8. If S_n is the set of n-element subsets of $\mathbb{Z}$ then S_n is countable since it is a subset of $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ (with n terms) and this set is countable. Then the set of finite subsets of $\mathbb{Z}$ is $\cup_{n=0}^{\infty} S_n$ which is a countable union of countable sets, hence countable.

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9. The functions $f : [1, 7) \to (2, 9)$ with $f(x) = 2 + (x/2)$ and $g : (2, 9) \to [1, 7)$ with $g(x) = 1 + (x/2)$ are both one-to-one, so by Cantor-Schröder-Bernstein there exists a bijection between $[1, 7)$ and $(2, 9)$.

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10. The functions $f : (0, 1) \to [0, 1]$ with $f(x) = x$ and $g : [0, 1] \to (0, 1)$ with $g(x) = (x + 1)/3$ are both one-to-one, so by Cantor-Schröder-Bernstein there exists a bijection between $(0, 1)$ and $[0, 1]$.
