E. Dummit's Math 4555 ∼ Complex Analysis, Fall 2022 ∼ Homework 9, due Fri Nov 11th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the radius of convergence for each power series centered at the given point:
   (a) The series expansion of $1/(z-2)$ centered at $z = 0$.
   (b) The series expansion of $z^3/(z^2+1)^2$ centered at $z = 1$.
   (c) The series expansion of $\sec z$ centered at $z = 0$.
   (d) The series expansion of $z \csc z$ centered at $z = 0$.
   (e) The series expansion of $\text{Log}(z)$ centered at $z = 1 + i$.
   (f) The series expansion of $\dfrac{1}{e^{1/z} - 1}$ centered at $z = i$.

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2. Solve the following optimization problems:
   (a) Find the maximum value of $\left|z^2 + 3z - 1\right|$ for $|z| \le 1$.
   (b) Find the maximum value of $\left|z^2 + i\right|$ for $|z| \le 2$.
   (c) [Optional] Find the maximum value of $|az^n + b|$ for $|z| \le r$, where r is a positive real constant and a, b are fixed complex numbers. [Hint: Triangle inequality.]

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Prove that the function $f(z)$ is entire if and only if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\lim_{n\to\infty} |a_n|^{1/n} = 0$.

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4. The goal of this problem is to prove the minimum modulus principle.
   (a) Suppose that $f(z)$ is holomorphic in a closed bounded region R and $|f(z)| > 0$ on R. Show that if the minimum value of $|f(z)|$ occurs at a point z_0 in the interior of R, then f is constant on R. [Hint: Consider $1/f$.]
   (b) Deduce that if $f(z)$ is holomorphic in a closed bounded region R and $|f(z)| > 0$ on R, then the minimum value of $|f(z)|$ on R must occur at a point on the boundary of R.
   (c) Show that the hypothesis $|f(z)| > 0$ cannot be removed from part (b) by giving an example of a nonconstant holomorphic $f(z)$ such that the minimum value of $|f(z)|$ occurs at a point z_0 in the interior of R.

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5. The goal of this problem is to give an estimate on the value of a monic polynomial. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and let $q(z) = z^n p(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + 1$ be its reverse polynomial.
   (a) Show that the maximum modulus of $q(z)$ on the circle $|z| = 1$ is at least 1, with equality if and only if $q(z)$ is the constant polynomial 1.
   (b) Show that the maximum modulus of $p(z)$ on the circle $|z| = 1$ is at least 1, with equality if and only if $p(z) = z^n$.

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6. The goal of this problem is to give another proof of the fundamental theorem of algebra, due to Boas. Suppose that $p(z) = \sum_{n=0}^{k} a_n z^n$ is a polynomial of degree $k \geq 1$ which is never zero.

   (a) Define $q(z) = p(z)\overline{p}(z) = [\sum_{n=0}^{k} a_n z^n][\sum_{n=0}^{k} \overline{a_n} z^n]$. Show that $q(z)$ has degree $2k \geq 2$, has real coefficients, and is never zero on $\mathbb{R}$ hence is either always positive or always negative on $\mathbb{R}$. [Hint: Notice that $\overline{p}(z) = \overline{p(\overline{z})}$.]

   (b) Continuing (a), let $r(z) = z^{2k} q(z + z^{-1})$. Show that $r(z)$ is holomorphic and nonzero.

   (c) Continuing (b), show that $-i \int_{\gamma} \frac{z^{2k-1}}{r(z)}\, dz = \int_{0}^{2\pi} \frac{1}{q(2\cos\theta)}\, d\theta$ where γ is the counterclockwise boundary of the unit circle. Explain why the first integral is zero while the second integral is nonzero, and obtain a contradiction.

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7. [Optional] The goal of this problem is to give another, another proof of the fundamental theorem of algebra that does not require any actual complex analysis. Suppose $p(z)$ is a polynomial.

   (a) Show that $|p(z)|$ must attain its minimum value at some point in $\mathbb{C}$. [Hint: Since $\lim_{|z|\to\infty} |p(z)| = \infty$, pick R with $|p(z)| > |p(0)|$ for $|z| > R$. Then use the extreme value theorem on the region $|z| \leq R$.]

   (b) Suppose that $q(z) = 1 + b(z - z_0)^r + \sum_{n=r+1}^{k} b_n (z - z_0)^n$ where $b \neq 0$. Show that there exists z with $|q(z)| < 1$. [Hint: Take $b(z - z_0)^r = -t$ and then show the sum is small relative to t as $t \to 0+$.]

   (c) Suppose that $p(z) = \sum_{n=0}^{k} a_n (z - z_0)^n$ is not constant and $|p(z_0)| > 0$. Show that there exists some z with $|p(z)| < |p(z_0)|$. [Hint: Write $p(z)/a_0 = 1 + b(z - z_0)^r + \sum_{n=r+1}^{k} b_n (z - z_0)^n$.]

   (d) Show that the minimum value of $|p(z)|$ must be zero and deduce that $p(z)$ has a root in $\mathbb{C}$.
