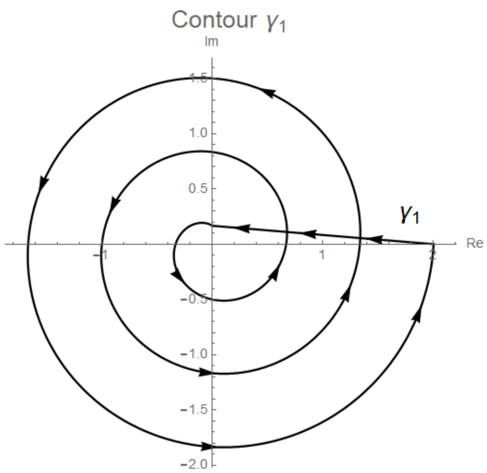
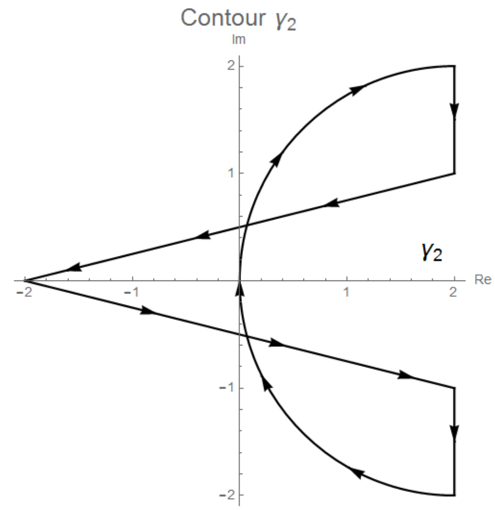
E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 8, due Fri Nov 4th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let γ_1 and γ_2 be the two contours plotted below. Find the requested quantities:





   (a) The winding numbers of γ_1 around 0, 1, i, $-i$, $1+i$, $2+i$, and $-2i$.

   (b) The winding numbers of γ_2 around -1, 1, $1+i$, $1-i$, and $-1+i$.

   (c) The values of $\int_{\gamma_1} \frac{1}{z-1}\,dz$ and $\int_{\gamma_2} \frac{1}{z-1}\,dz$.

   (d) The values of $\int_{\gamma_1} \frac{1}{z-(1+i)}\,dz$ and $\int_{\gamma_2} \frac{1}{z-(1+i)}\,dz$.

   (e) The values of $\int_{\gamma_1} \frac{e^z}{z-1}\,dz$ and $\int_{\gamma_2} \frac{e^z}{z-1}\,dz$.

   (f) The value of $\int_{\gamma_1} \frac{2z}{z^2-1}\,dz$.

   (g) The value of $\int_{\gamma_2} \frac{2}{z^2-2z+2}\,dz$.

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2. For each function f on each contour γ, calculate $\int_\gamma f(z)\,dz$:

   (a) $f(z) = \frac{e^z}{z-2}$ on the counterclockwise boundary of the circle $|z| = 4$.

   (b) $f(z) = \frac{\sin^2(2z)}{z-1}$ on the counterclockwise boundary of the circle $|z| = 4$.

   (c) $f(z) = \frac{z^2+1}{z^2-1}$ on the counterclockwise boundary of the square with vertices ± 10, $\pm 10i$.

   (d) $f(z) = \frac{z^2+1}{z^2-1}$ on the counterclockwise boundary of the square with vertices 0, $10-10i$, 20, $10+10i$.

   (e) $f(z) = \frac{e^z}{z \sin z}$ on the counterclockwise boundary of the circle $|z-\pi| = e^{-\pi}$.

   (f) $f(z) = \overline{z}e^{1/\overline{z}}$ on the counterclockwise boundary of the circle $|z| = 2$. [Hint: On γ, $\overline{z}$ can be written in terms of z.]

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3. If γ is the unit circle traversed once counterclockwise, let $I(a,b) = \int_\gamma \frac{1}{(z-a)(z-b)}\, dz$.

   (a) Find $I(a,a)$ if $|a| \neq 1$.
   (b) Find $I(a,b)$ if $|a| < 1$ and $|b| < 1$ and $a \neq b$.
   (c) Find $I(a,b)$ if $|a| < 1$ and $|b| > 1$.
   (d) Find $I(a,b)$ if $|a| > 1$ and $|b| > 1$ and $a \neq b$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. Suppose $p(z)$ is a polynomial of degree at least 2. Then $p(z)$ has finitely many complex zeroes, so they are all contained in a disc $|z| < r$ for some r. The goal of this problem is to prove that $\int_\gamma \frac{1}{p(z)}\, dz = 0$ where γ is the circle $|z| = r$ traversed once counterclockwise. More generally, let $I(R) = \int_{\gamma_R} \frac{1}{p(z)}\, dz$ where γ_R is the circle $|z| = R$ traversed once counterclockwise.

   (a) Show that $I(R) = I(r)$ for all $R \geq r$.
   (b) Show that $|I(R)| \leq \frac{2\pi R}{|a|\,(R-r)^d}$ where p has degree d and leading coefficient a. [Hint: You may assume $p(z)$ factors as $p(z) = a(z-z_1)(z-z_2)\cdots(z-z_d)$.]
   (c) Show that $\lim_{R\to\infty} I(R) = 0$. Deduce that $I(r) = 0$.

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5. The goal of this problem is to give another proof of the differentiation-via-integration formula $f'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}\, dz$. So suppose f is a holomorphic function on a simply connected region R and let γ be a counterclockwise circle of radius $r > 0$ centered at z_0 in the interior of R such that the disc $|z - z_0| \leq r$ lies inside R.

   (a) Show that $\frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)(z-z_0-h)}\, dz$. [Hint: Use Cauchy's integral formula.]
   (b) For $|h| < r$, let $g_h(z) = \frac{f(z)}{(z-z_0)(z-z_0-h)}$. Show that as $h \to 0$ the functions $g_h(z)$ converge uniformly to the limit $g(z) = \frac{f(z)}{(z-z_0)^2}$. [Hint: Restrict attention to $|h| < r/2$, then suppose $|f(z)| \leq M$ on γ_r and bound $|g_h(z) - g(z)|$ from above.]
   (c) Show that $f'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}\, dz$. [Hint: Use uniform convergence to change the order of the integral and the limit as $h \to 0$.]

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6. The goal of this problem is to establish some basic facts about the gamma and zeta functions, which are two special functions with broad utility in complex analysis, number theory, statistics, physics, and various other areas. Recall that for a positive real number α and a complex number z, we have $\alpha^z = e^{z \ln \alpha}$.

   (a) Suppose n is a positive integer. Show that $\int_0^\infty t^{n-1} e^{-t}\, dt = (n-1)!$.
   (b) Suppose $\mathrm{Re}(z) > 1$. Show that the integral $\int_0^\infty t^{z-1} e^{-t}\, dt$ converges absolutely. [Hint: First do $z \in \mathbb{R}$.]
   (c) Let R be the region with $\mathrm{Re}(z) > 1$. Show that $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is holomorphic on R. [Hint: Differentiate under the integral sign.]
   (d) Suppose $\mathrm{Re}(z) > 1$. Show that the series $\sum_{n=1}^\infty \frac{1}{n^z}$ converges absolutely. [Hint: Integral comparison.]
   (e) Let R be the region with $\mathrm{Re}(z) > 1$. Show that $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$ is holomorphic on R.
