E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 7, due Wed Oct 26th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each function f on each contour γ, calculate $\int_\gamma f(z)\,dz$ without[1] using a parametrization of γ:
   (a) $f(z) = z^{-3}$ on the upper half of the unit circle traversed from $z = 1$ to $z = -1$.
   (b) $f(z) = z^n$ (n an integer with $n \neq -1$) on the counterclockwise boundary of the square with vertices ± 1, $\pm i$.
   (c) $f(z) = 3z^2$ on the curve $\gamma(t) = e^{e^t} - \tan(t^2)i$ for $0 \le t \le 1$.
   (d) $f(z) = e^z$ on the portion of the ellipse $\dfrac{\mathrm{Re}(z)^2}{(\ln 2)^2} + \dfrac{\mathrm{Im}(z)^2}{4\pi^2} = 1$ clockwise from $z = \ln 2$ to $z = 2\pi i$.
   (e) $f(z) = \mathrm{Log}(z)$ on the polygonal path with successive vertices i, $-1 + 7i$, $-20 - 22i$, and $-i$.

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2. For each function f on each closed contour γ, calculate $\int_\gamma f(z)\,dz$:
   (a) $f(z) = z^{-1}$ on the counterclockwise boundary of the square with vertices 1 and $-2 \pm i\sqrt{3}$.
   (b) $f(z) = z^{-1}$ on the counterclockwise boundary of the triangle with vertices -1 and $-2 \pm i\sqrt{3}$.
   (c) $f(z) = z^{-1}$ on the polygonal path with successive vertices 1, i, -1, $-i$, 2, $2i$, -2, $-2i$, and 1.
   (d) $f(z) = \dfrac{i}{z^2 - iz}$ on the counterclockwise boundary of the ellipse $\dfrac{\mathrm{Re}(z)^2}{(\ln 2)^2} + \dfrac{\mathrm{Im}(z)^2}{4\pi^2} = 1$.
   (e) $f(z) = \dfrac{z}{z^2 + 1}$ on the counterclockwise boundary of the square with vertices ± 1 and $\pm 1 + 2i$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Recall that if $\gamma_0 : [0, 1] \to \mathbb{C}$ and $\gamma_1 : [0, 1] \to \mathbb{C}$ are continuous curves in a region R, we say they are homotopic in R if there exists some continuous function $h : [0, 1] \times [0, 1] \to R$ with $h(s, 0) = h(s, 1)$ for all s and $h(0, t) = \gamma_0(t)$ and $h(1, t) = \gamma_1(t)$. Prove that being homotopic in R is an equivalence relation on continuous curves in R. [Hint: For transitivity, use half of the interval to go from γ_0 to γ_1 and the other half to go from γ_1 to γ_2.]

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4. We have previously seen that a holomorphic function on a region R is one with $\dfrac{\partial f}{\partial \overline{z}} = 0$. We say a function f is antiholomorphic on R if $\dfrac{\partial f}{\partial z} = 0$.
   (a) Which of z, $\overline{z}$, $z\overline{z}$, $\dfrac{1}{z}$, $\dfrac{z}{\overline{z}}$, e^z, $e^{\overline{z}}$, and $\overline{e^z}$ are antiholomorphic?
   (b) Prove that f is antiholomorphic on R if and only if $\overline{f}$ is holomorphic on R. [Hint: Calculate everything in real and imaginary parts.]
   (c) Show that the only holomorphic antiholomorphic functions on R are constants.

[1] Obviously, since this is in Part I, I won't actually know for sure that you didn't use a parametrization, but read between the lines on the directions here.

(d) If f is antiholomorphic, show that f reverses angles, in the sense that if f is antiholomorphic on R and two differentiable curves c_1 and c_2 pass through the point $z_0 \in R$ with $\frac{\partial f}{\partial \overline{z}}(z_0) \neq 0$, then the angle between c_1 and c_2 at z_0 is the negative of the angle between $f \circ c_1$ and $f \circ c_2$ at $f(z_0)$. [Hint: Use (b).]

Next, we extend this notion to integrals $\int_\gamma f(z)\, d\overline{z}$ with differential $d\overline{z}$. Formally, if P^* is a tagged partition of $[a, b]$, then in place of the Riemann sum $\sum_{i=1}^n f(z_i^*)[z_i - z_{i-1}]$ for $\int_\gamma f(z)\, dz$, we use the Riemann sum $\sum_{i=1}^n f(z_i^*)[\overline{z_i} - \overline{z_{i-1}}]$ for $\int_\gamma f(z)\, d\overline{z}$. More conveniently, if we write $dz = dx + i\, dy$ and $d\overline{z} = dx - i\, dy$, then we have the formulas $\int_\gamma f(z)\, dz = \int_\gamma f(z)\, dx + i \int_\gamma f(z)\, dy$ and $\int_\gamma f(z)\, d\overline{z} = \int_\gamma f(z)\, dx - i \int_\gamma f(z)\, dy$.

(e) If f is integrable on the contour γ, show that $\overline{\int_\gamma f(z)\, dz} = \int_\gamma \overline{f(z)}\, d\overline{z}$.

(f) Prove Cauchy's integral theorem for antiholomorphic functions: if f is antiholomorphic on a simply connected region R and γ is a closed contour in R, then $\int_\gamma f\, d\overline{z} = 0$.

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5. The goal of this problem is to give another approach for evaluating the integral $I_\gamma(z_0) = \int_\gamma \frac{1}{z - z_0}\, dz$ where γ is any counterclockwise-oriented circle not containing z_0, which was the subject of problem 5 of homework 6.

(a) Suppose z_0 is a distance $R > 0$ away from the closest point on the circle. Show that $\left| \int_\gamma \frac{1}{z - z_0}\, dz \right| \leq \frac{2\pi r}{R}$ where r is the radius of the circle. [Hint: Bound the integral by the arclength times the maximum of the function.]

(b) Suppose z_0 is outside the circle. Show that $\int_\gamma \frac{1}{z - z_0}\, dz = 0$. [Hint: Move γ far away from z_0.]

(c) Suppose z_0 is in the interior of the circle. Show that $\int_\gamma \frac{1}{z - z_0}\, dz = 2\pi i$. [Hint: Recenter γ at z_0.]

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6. The goal of this problem is to give a third approach for evaluating the integral $I_\gamma(z_0) = \int_\gamma \frac{1}{z - z_0}\, dz$ where γ is any counterclockwise-oriented circle not containing z_0.

(a) Suppose z_0 is outside the circle. Show that $\int_\gamma \frac{1}{z - z_0}\, dz = 0$. [Hint: Pick a branch of $\log(z - z_0)$ that does not intersect the circle.]

(b) Suppose z_0 is in the interior of the circle and let the horizontal ray $z = z_0 + t$ for $t \geq 0$ intersect the circle at P. Choose any α and β on the circle such that the points P, α, β are in counterclockwise order around the circle, and take $\tilde{\gamma}$ to be the counterclockwise arc from α to β. Show that $\int_{\tilde{\gamma}} \frac{1}{z - z_0}\, dz = \operatorname{Log}(\beta - z_0) - \operatorname{Log}(\alpha - z_0)$.

(c) Suppose z_0 is in the interior of the circle. Show that $\int_\gamma \frac{1}{z - z_0}\, dz = 2\pi i$. [Hint: In (b), let α approach P from above and β approach P from below.]
