E. Dummit's Math 4555 $\sim$ Complex Analysis, Fall 2022 $\sim$ Homework 6, due Wed Oct 19th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each function f on each curve γ, calculate $\int_\gamma f(z)\,dz$:

   (a) $f(z) = 1/z$, $\gamma(t) = e^{it}$ for $0 \le t \le \pi/2$.

   (b) $f(z) = z^n$ (n an integer with $n \ne -1$), $\gamma(t) = e^{it}$ for $0 \le t \le \pi/2$.

   (c) $f(z) = 3z^2$, $\gamma(t) = t + (1-t)i$ for $0 \le t \le 1$.

   (d) $f(z) = 3\overline{z}^2$, $\gamma(t) = t + (1-t)i$ for $0 \le t \le 1$.

   (e) $f(z) = \mathrm{Log}(z)$, $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$.

---

2. For each contour γ, give a parametrization and then evaluate $\int_\gamma f(z)\,dz$:

   (a) The circle $|z-1| = 2$ traversed twice counterclockwise, where $f(z) = \overline{z}$.

   (b) The counterclockwise boundary of the ellipse $\dfrac{\mathrm{Re}(z)^2}{a^2} + \dfrac{\mathrm{Im}(z)^2}{b^2} = 1$, where $f(z) = \mathrm{Re}(z)$ and $a, b > 0$.

   (c) The counterclockwise boundary of the triangle with vertices 0, 2, and $1+i$, where $f(z) = \mathrm{Re}(z)$.

   (d) The counterclockwise boundary of the square with vertices 0, 1, $1+i$, and i, where $f(z) = z$.

---

3. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a contour. The <u>length</u> of the contour γ is defined as $\int_a^b |\gamma'(t)|\,dt$. Note that for $\gamma(t) = x(t) + iy(t)$ this formula is merely the familiar arclength formula $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$.

   (a) Find the length of the contour parametrized by $\gamma(t) = 4e^{3it}$, $0 \le t \le \pi$.

   (b) Find the length of the contour parametrized by $\gamma(t) = t + i\cosh(t)$, $0 \le t \le 1$.

   (c) Find the length of the contour parametrized by $\gamma(t) = (1-2t) + (3+t)i$, $0 \le t \le 1$.

   (d) Find the length of the contour parametrized by $\gamma(t) = (\pi + et) + (\sqrt{2} - \pi t)i$, $0 \le t \le 1$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. Let $f(z)$ be continuous on a connected open region R. Prove that the following are equivalent (you may freely appeal to theorems proven in the notes):

   (a) The line integral $\int_\gamma f(z)\,dz = 0$ for all closed contours γ in R.

   (b) There exists a holomorphic $F(z)$ on R such that $F'(z) = f(z)$.

   (c) For any $a, b \in R$ and any γ_1 and γ_2 contours in R from a to b, $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

---

5. The goal of this problem is to compute the integral $I(z_0) = \int_\gamma \frac{dz}{z - z_0}$ where γ is the unit circle traversed once counterclockwise and $|z_0| \neq 1$.

   (a) If $z_0 = re^{i\theta}$ show that $I(z_0) = I(r)$. [Hint: Make a substitution.]

   (b) Suppose $r \geq 0$ and $r \neq 1$. Show that $\int_0^{2\pi} \frac{r \sin t}{1 - 2r \cos t + r^2}\, dt = 0$.

   (c) Show that $\int_0^{\pi} \frac{1 - r \cos t}{1 - 2r \cos t + r^2}\, dt = \begin{cases} \pi & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r > 1 \end{cases}$. [Hint: Substitute $x = \tan(t/2)$, which has $dt = \frac{dx}{1 + x^2}$ and $\cos t = \frac{1 - x^2}{1 + x^2}$. You may want to make a computer do the partial fraction decomposition.]

   (d) Find the value of $I(z_0)$ in terms of z_0.

---

6. The goal of this problem is to give an integral where evaluating it using Riemann sums is actually easier than most other approaches. Fix $a > 1$ and consider the integral $I_a = \int_0^{\pi} \ln(a^2 - 2a \cos x + 1)\, dx$.

   (a) Show that $\prod_{k=1}^{n-1} (a^2 - 2a \cos \frac{k\pi}{n} + 1) = \frac{a^{2n} - 1}{a^2 - 1}$. [Hint: Factor over $\mathbb{C}$.]

   (b) Show that the right-endpoint Riemann sum for I_a with partition $P = \{0, \pi/n, 2\pi/n, \ldots, \pi(n-1)/n, \pi\}$ is equal to $\frac{\pi}{n} \ln \frac{a^{2n} - 1}{a^2 - 1} + \frac{\pi}{n} \ln(a + 1)^2$.

   (c) Show that the value of the integral I_a is $2\pi \ln a$. [Hint: The expression from (b) equals $\frac{\pi}{n} \ln(a^{2n} - 1) + \frac{\pi}{n} \ln \frac{a + 1}{a - 1}$.]

---

7. [Optional] Recall the definition of the length of a contour from problem 3.

   (a) Suppose that $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are continuously differentiable and there exists a continuously differentiable increasing function $g : [a, b] \to [c, d]$ with $g(a) = c$ and $g(b) = d$ such that $\gamma_1 = \gamma_2 \circ g$, show that $\int_a^b |\gamma_1'(t)|\, dt = \int_c^d |\gamma_2'(s)|\, ds$. [Hint: Substitution.]

   (b) Prove that the length of a contour is independent of the parametrization. [Hint: Sum (a).]

   (c) Suppose that $|f(z)| \leq M$ on the contour γ of length s. Prove that $\left|\int_\gamma f(z)\, dz\right| \leq Ms$. [Hint: Riemann sums. You may assume that the length of a polygonal path connecting successive points on γ converges to the length of γ as the norm of the path approaches 0.]
