E. Dummit's Math 4555 ∼ Complex Analysis, Fall 2022 ∼ Homework 5, due Wed Oct 12th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find all solutions $z \in \mathbb{C}$ to each of the following equations:
   (a) $e^{4z} = i$.
   (b) $e^{iz} = 4$.
   (c) $\cosh(z) = 5/4$.
   (d) $\cos(z) = 5/4$.
   (e) $\sinh(z) = i\cosh(z)$.
   (f) $\sinh(z) = \cosh(z)$.
   (g) $\sin(z) = i\cos(z)$.
   (h) $\sinh(z) = (1+3i)/4$.

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2. Find all possible complex values for each of these expressions (note that log always denotes the multivalued complex logarithm):
   (a) $\log(i)$.
   (b) i^{2i}.
   (c) 4^i.
   (d) $e^{\log(i)}$.
   (e) $\log(e^i)$.
   (f) $1^{1/6}$.
   (g) $[1^{1/6}]^2$.
   (h) $[1^2]^{1/6}$.
   (i) $(-1)^{1/\pi}$.
   (j) $\log(\log(i))$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. By differentiating a power series expansion, we can often show it satisfies a differential equation inside its radius of convergence. Show the following:
   (a) Show that $f(z) = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!}$ satisfies $f''(z) = f(z)$ for all $z \in \mathbb{C}$.
   (b) Show that $f(z) = \sum_{n=1}^{\infty} \dfrac{z^n}{n}$ satisfies $f'(z) = \dfrac{1}{1-z}$ for $|z| < 1$.
   (c) Show that $f(z) = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(n!)^2}$ satisfies $z^2 f''(z) + zf'(z) = 4z^2 f(z)$ for all $z \in \mathbb{C}$.

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4. We can often construct series solutions to a differential equation involving a function $f(z)$ by writing $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, differentiating formally, and then solving for the coefficients a_i. An advantage to this approach is that it can also provide an easy proof for uniqueness of the solution under the (very reasonable) assumption that the solution is analytic at $z = z_0$.
   (a) Show that the unique analytic solution at $z = 0$ to $f'(z) = f(z)$ with $f(0) = 1$ is $f(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$.
   (b) Find the terms up to order 4 in a series expansion for an analytic solution at $z = 0$ to the differential equation $f''(z) + 3zf(z) = 1$ with $f(0) = f'(0) = 1$.
   (c) Find the terms up to order 4 in a series expansion for an analytic solution at $z = 1$ to the differential equation $f''(z) + f'(z) + (z-1)^2 f(z) = 0$ with $f(1) = f'(1) = 1$.

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5. The goal of this problem is to prove that if p is any polynomial, then the power series $\sum_{n=0}^{\infty} p(n) z^n$ is a rational function in z.
   (a) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $zf'(z) = \sum_{n=0}^{\infty} n a_n z^n$.
   (b) Show that for every integer $k \geq 0$, $\sum_{n=0}^{\infty} n^k z^n$ is a rational function in z. [Hint: Induct on k.]
   (c) Show that $\sum_{n=0}^{\infty} p(n) z^n$ is a rational function in z for any polynomial p.
   (d) Express $\sum_{n=0}^{\infty} (2n+5) z^n$ and $\sum_{n=0}^{\infty} (n^2+n) z^n$ as rational functions in z.

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6. The goal of this problem is to prove (versions of) L'Hôpital's Rule for $\frac{0}{0}$ limits of holomorphic and analytic functions.

   (a) Suppose that f and g are holomorphic at $z = a$, and that $f(a) = g(a) = 0$ where $g'(a) \neq 0$. Prove that $\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}$. [Hint: Evaluate $\frac{[f(z) - f(a)]/(z-a)}{[g(z) - g(a)]/(z-a)}$.]

   (b) Suppose that f and g are analytic at $z = a$ and that $f(a) = g(a) = 0$ where g has order d at $z = a$ (i.e., the first nonzero coefficient in the series expansion for g is the coefficient of $(z-a)^d$). Prove that $\lim_{z\to a} \frac{f(z)}{g(z)}$ exists if and only if f has order at least d at $z = a$, and in such a case, $\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f^{(d)}(a)}{g^{(d)}(a)}$. [Hint: Restrict attention to a disc of positive radius where f and g have their only zero at $z = a$.]
