E. Dummit's Math 4555 ∼ Complex Analysis, Fall 2022 ∼ Homework 4, due Wed Oct 5th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the radius and the disc of convergence for each power series:
   (a) $\sum_{n=0}^{\infty}(z-1+i)^n$.
   (b) $\sum_{n=0}^{\infty}\dfrac{(z-i)^n}{n!}$.
   (c) $\sum_{n=1}^{\infty} n^n(z-1)^n$.
   (d) $\sum_{n=1}^{\infty}\dfrac{(z+2)^n}{n^n}$.
   (e) $\sum_{n=0}^{\infty}(2z+1)^n$.
   (f) $\sum_{n=1}^{\infty}\dfrac{\pi^n}{n^e}(\pi z+e)^n$.
   (g) $\sum_{n=0}^{\infty}\cosh(n)z^n$.
   (h) $\sum_{n=0}^{\infty}\dfrac{(3z+i)^{3n}}{(2-i)^n}$.

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2. Find power series expansions for each given function $f(z)$ centered at the given point $z = z_0$:
   (a) $f(z) = \dfrac{z}{1-z^3}$ around $z = 0$. [Hint: Use $\dfrac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.]
   (b) $f(z) = 1 + z + z^2 + z^4$ around $z = -2$.
   (c) $f(z) = (1 + z)/(1 - z)$ around $z = -1$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Prove the following things about the complex exponential and (hyperbolic) trigonometric functions:
   (a) Show $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$ and $\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$.
   (b) Show $\sinh(z+w) = \sinh(z)\cosh(w)+\cosh(z)\sinh(w)$ and $\cosh(z+w) = \cosh(z)\cosh(w)+\sinh(z)\sinh(w)$.
   (c) Show $\tanh(z + w) = \dfrac{\tanh(z) + \tanh(w)}{1 + \tanh(z)\tanh(w)}$. Deduce that $\tanh(z)$ is periodic with period $i\pi$.
   (d) Show e^z is one-to-one on any open disc of radius π.
   (e) Show $2\cos(\frac{z+w}{2})\sin(\frac{z-w}{2}) = \sin(z) - \sin(w)$. Deduce that $\sin(z) = \sin(w)$ if and only if $z + w = (2k + 1)\pi$ or $z - w = 2k\pi$ for an integer k.
   (f) Show $\sin(z)$ is one-to-one on the region $-\pi < \mathrm{Re}(z) < \pi$, $\mathrm{Im}(z) > 0$.

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4. The goal of this problem is to study the complex analogue of Newton's binomial series. Let α be any complex number that is not a nonnegative integer. Define the binomial coefficient $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$ for each integer $n \geq 0$. Now define the binomial series $B_\alpha(z) = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n$. In 1665, Newton proved that if α is real, then $B_\alpha(x) = (1 + x)^\alpha$ for all real $-1 < x < 1$.
   (a) Show that the radius of convergence of the binomial series equals 1. [Hint: Use the Ratio Test.]
   (b) Show that $(B_{1/m}(z))^m = 1+z$ for all $|z| < 1$. [Hint: Use Newton's binomial theorem and the uniqueness of series expansions.]
   (c) Deduce that for $|z| < 1$, the binomial series $B_{1/m}(z)$ is a holomorphic function of z whose value is an mth root of $1 + z$.

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5. Suppose f is a power series centered at z_0 with radius of convergence R_f and g is a power series centered at z_0 with radius of convergence R_g, where $R_f \leq R_g$.

   (a) Show that $f + g$ has radius of convergence at least R_f.

   (b) If $R_f < R_g$ show that $f + g$ has radius of convergence exactly R_f.

   (c) Find an example of power series f and g with $R_f = R_g$ such that $f + g$ has radius of convergence strictly greater than R_f.

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6. Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. The goal of this problem is to study the power series $f(z) = \sum_{n=0}^{\infty} F_n z^n$, the generating function for the Fibonacci numbers.

   (a) Show that $(1 - z - z^2)f(z) = z$ as a formal power series and deduce $f(z) = \dfrac{z}{1 - z - z^2}$.

   (b) Find complex constants a, α, b, β such that $\dfrac{z}{1 - z - z^2} = \dfrac{a}{1 - \alpha z} + \dfrac{b}{1 - \beta z}$.

   (c) Prove Binet's formula for the Fibonacci numbers: $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$ where $\varphi = \dfrac{1 + \sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1 - \sqrt{5}}{2}$.
   [Hint: Expand the two geometric series from (b) and compare to $f(z)$.]

   (d) Find the radius of convergence of $f(z)$. [Hint: Use problem 5(b).]

   - Remark: A similar method to the one in (a)-(c) can be used to solve any linear recurrence with constant coefficients, of the form $a_{n+1} = c_n a_n + \cdots + c_{n-k} a_{n-k}$ for constants c_i.
