E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 3, due Wed Sep 28th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For f, g, h as below, perform the requested calculations:

$$\begin{aligned} f &= \sum_{n=0}^{\infty} n^2 z^n &&= z + 4z^2 + 9z^3 + 16z^4 + 25z^5 + \cdots \\ g &= \sum_{n=0}^{\infty} (1+2n) z^n &&= 1 + 3z + 5z^2 + 7z^3 + 9z^4 + 11z^5 + \cdots \\ h &= \sum_{n=-2}^{\infty} (-1)^n z^n &&= z^{-2} - z^{-1} + 1 - z + z^2 - z^3 + z^4 - \cdots \end{aligned}$$

   (a) Find $f + 2g$ up through the terms of order 4.
   (b) Find $f + h$ up through the terms of order 4.
   (c) Find fg up through the terms of order 4.
   (d) Find fh up through the terms of order 3.
   (e) Find f^2 up through the terms of order 5.
   (f) Find g^{-1} up through the terms of order 4.
   (g) Find f^{-1} up through the terms of order 3.
   (h) Find h^{-1}. [Hint: It is a polynomial.]
   (i) Find f/g up through the terms of order 3.

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2. Find the radius of convergence for each power series:

   (a) $\sum_{n=0}^{\infty} \dfrac{2^n}{n+2} z^n$.
   (b) $\sum_{n=1}^{\infty} \dfrac{z^n}{n^n}$.
   (c) $\sum_{n=1}^{\infty} n^n z^n$.
   (d) $\sum_{n=0}^{\infty} \dfrac{(3+2i)^n}{(1-i)^n} z^n$.
   (e) $\sum_{n=0}^{\infty} (i - \dfrac{1}{n})^n z^n$.
   (f) $\sum_{n=0}^{\infty} \dfrac{2^n}{3^n + 4^n} z^n$.
   (g) $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n+1}$.
   (h) $\sum_{n=0}^{\infty} z^{2^n}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. The goal of this problem is to prove the Ratio Test. So suppose $\{a_n\}_{n \geq 1}$ is a complex sequence whose terms are nonzero.

   (a) Suppose that there exists a real number $\rho < 1$ and positive N such that $|a_{n+1}/a_n| \leq \rho$ for all $n \geq N$. Show that $\sum_{n=1}^{\infty} a_n$ converges absolutely. [Hint: Show that $|a_{N+k}| \leq |a_N| \rho^k$ for all $k \geq 0$ and use this to show that $\sum_{n=N}^{\infty} |a_n|$ is finite.]
   (b) Show that if we weaken the hypothesis of (a) only to require that $|a_{n+1}/a_n| < 1$ for all n, then $\sum_{n=1}^{\infty} a_n$ need not converge absolutely.
   (c) Suppose that there exists a positive N such that $|a_{n+1}/a_n| \geq 1$ for all $n \geq N$. Show that $\sum_{n=1}^{\infty} a_n$ diverges. [Hint: The terms cannot go to zero.]
   (d) Prove the Ratio Test: If $\lim_{n\to\infty} |a_{n+1}/a_n| = r$ exists, then $\sum_{n=1}^{\infty} a_n$ converges absolutely if $r < 1$ and $\sum_{n=1}^{\infty} a_n$ diverges if $r > 1$. [Hint: For the convergence, take any ρ with $r < \rho < 1$ in (a).]
   (e) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \dfrac{n^n}{n!} z^n$.

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4. The goal of this problem is to discuss a useful summation technique known as Abel summation (also called summation by parts), and then use it to derive some series convergence tests. So suppose $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two complex sequences, and define $S_n = \sum_{k=1}^{n} a_k b_k$ and $B_n = \sum_{k=1}^{n} b_k$.

   (a) Show the Abel summation formula: that $S_n = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$.

   (b) Prove Dirichlet's convergence test: if $\{a_n\}_{n\geq 1}$ is a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and the sequence $\{B_n\}_{n\geq 1}$ is bounded, then $\sum_{k=1}^{\infty} a_k b_k$ converges. [Hint: Suppose $|B_n| \leq M$ for all n. Use (a) on the partial sum S_n and then show that $a_n B_n \to 0$ and that $\sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$ is absolutely convergent.]

   (c) Deduce the Alternating Series Test: if $\{a_n\}_{n\geq 1}$ is a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

   - Remark: The summation formula in part (a) is the discrete analogue of integration by parts, whence its name "summation by parts". Specifically, the sum $\sum_{k=1}^{n} f(k)$ is the analogue of the antiderivative $\int f(x)\,dx$ while the difference $f(k+1) - f(k)$ is the analogue of the derivative $f'(x)$. The formula in (a) is then the analogue of the integration by parts formula $\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$ where G is an antiderivative of g (the minus sign arises because the differences in (a) are $a_k - a_{k+1}$ rather than $a_{k+1} - a_k$).

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5. The goal of this problem is to study the convergence of the innocuous-seeming power series $f = \sum_{n=1}^{\infty} \frac{z^n}{n}$.

   (a) Show that f has radius of convergence 1 and that f does not converge for $z = 1$.

   (b) Show that if $0 < \theta < 2\pi$, then $\left|\sum_{k=0}^{n} e^{ik\theta}\right| \leq \frac{2}{|e^{i\theta} - 1|}$. [Hint: Use problem 6(d) of homework 1.]

   (c) Show that f does converge if $|z| = 1$ but $z \neq 1$. [Hint: Use (b) with $z = e^{i\theta}$ in Dirichlet's convergence test from problem 4(b).]

   (d) Determine all z for which $f = \sum_{n=0}^{\infty} \frac{z^n}{n}$ converges.

   (e) Determine all z for which $\sum_{n=1}^{\infty} \frac{z^{2n}}{n}$ converges. [Hint: Let $t = z^2$.]
