E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 2, due Wed Sep 21st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Plot each of the given regions in the complex plane (you may want to use a computer). For each region, identify whether it is (i) open, (ii) closed, (iii) connected, and (iv) bounded.

   (a) $|z-1| < 1$.
   (b) $1 \le |z| \le 3$.
   (c) $|z| > |z-1|$.
   (d) $0 < \text{Im}(z) \le 1$.
   (e) $\text{Re(z)} \le \text{Im(z)}$.
   (f) $\text{Re}(z) \cdot \text{Im}(z) > 1$.
   (g) $|z(z-2)| \le 1$.
   (h) $|z(z-4)| \le 1$.

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2. Compute the following complex limits or show they do not exist:

   (a) $\lim_{z \to 1} \dfrac{z^3-1}{z-1}$.
   (b) $\lim_{z \to i} \dfrac{z^3-1}{z-1}$.
   (c) $\lim_{z \to i} \dfrac{z}{|z|}$.
   (d) $\lim_{z \to 0} \dfrac{z}{|z|}$.
   (e) $\lim_{z \to 0} \dfrac{z^3}{|z|^2}$.
   (f) $\lim_{z \to 1} \dfrac{z^2-1}{z^2+z-2\overline{z}}$.

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3. For each complex function, calculate its partial derivatives $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \overline{z}}$, and determine whether the complex derivative f' exists on any open region R.

   (a) $f(z) = z^4 + z$.
   (b) $f(z) = 3z\overline{z}^2 + z^4$.
   (c) $f(z) = \dfrac{e^z}{\overline{z}-1}$.

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4. For each complex function, calculate its partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, and determine whether the complex derivative f' exists using the Cauchy-Riemann equations.

   (a) $f(x+iy) = (2x^2+y) + (2y^2-x)i$.
   (b) $f(x+iy) = 4xy + (2y^2-2x^2)i$.
   (c) $f(x+iy) = (3 + e^y \sin x) - (e^y \cos x)i$.
   (d) $f(x+iy) = \sin x \cos y - i \cos x \sin y$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Suppose that we define two differential operators $L = a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y}$ and $M = c\dfrac{\partial}{\partial x} + d\dfrac{\partial}{\partial y}$ for some constants $a, b, c, d \in \mathbb{C}$, meaning that $Lf = a\dfrac{\partial f}{\partial x} + b\dfrac{\partial f}{\partial y}$ and similarly $Mf = c\dfrac{\partial f}{\partial x} + d\dfrac{\partial f}{\partial y}$ for a function f. Show that if $Lz = 1$, $L\overline{z} = 0$, $Mz = 0$, and $M\overline{z} = 1$ for all $z, \overline{z}$, then in fact we must have $a = \dfrac{1}{2}$, $b = -\dfrac{i}{2}$, $c = \dfrac{1}{2}$, $d = \dfrac{i}{2}$ so that $L = \dfrac{\partial}{\partial z}$ and $M = \dfrac{\partial}{\partial \overline{z}}$.

   - Remark: The point of this calculation is that our definitions of $L = \dfrac{\partial}{\partial z}$ and $M = \dfrac{\partial}{\partial \overline{z}}$ are forced to be the ones we selected if we want them to act on z and $\overline{z}$ in the expected way.

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6. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is twice differentiable. We define the Laplacian of f to be $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$, and we say f is harmonic if $\Delta f = 0$ on the entire domain of f.

   (a) Find the Laplacians of $3x - y$, $x^2 - y^2$, e^{x+y}, $e^x \cos y$, $e^y \cos x$, $\dfrac{1}{x^2 + y^2}$, $\ln(x^2 + y^2)$, and $\tan^{-1}(y/x)$. Which of these are harmonic?

   (b) Suppose $h(z) = f(x, y) + ig(x, y)$ is a function of $z = x + iy$ where f and g are both twice continuously differentiable. Show that $4\dfrac{\partial^2 f}{\partial z \partial \overline{z}} = \Delta f$. [Hint: Partial derivatives can be interchanged for twice continuously differentiable functions.]

   (c) Suppose $h(z) = f(x, y) + ig(x, y)$ is a holomorphic function of $z = x + iy$ on the region R. Show that f and g are harmonic on R.

   Part (c) shows that the real and imaginary parts of a holomorphic function are harmonic. As we will show later, the converse is also broadly true: a harmonic function defined on a sufficiently nice region is necessarily the real (or imaginary) part of a holomorphic function. Our goal now is to establish this result explicitly for polynomials.

   (d) Suppose that $p(x, y)$ is a real-valued harmonic polynomial in x and y. Show that p must be the sum of a polynomial in z with a polynomial in $\overline{z}$. [Hint: Write $p = \sum_{i,j \geq 0} a_{i,j} z^i \overline{z}^j$ and use (b) to show that $ija_{i,j}$ must equal zero for all i, j. You may use the fact that the only polynomial that is identically zero on all of $\mathbb{C}$ is the zero polynomial.]

   (e) Suppose that $p(z, \overline{z}) = c + \sum_{i=1}^{n} (a_i z^i + b_i \overline{z}^i)$ for some $c, a_i, b_i \in \mathbb{C}$ and that p is real-valued. Show that c is real and that $a_i = \overline{b_i}$ for each i, and deduce that $p = \text{Re}[c + \sum_{i=1}^{n} a_i z^i]$ so that p is the real part of a holomorphic function. [Hint: Compute $p - \overline{p}$.]

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7. We have already discussed how to convert between the "rectangular" differential operators $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}$ and the "complex" differential operators $\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial \overline{z}}$. The goal of this problem is to write down the "polar" differential operators $\dfrac{\partial}{\partial r}$ and $\dfrac{\partial}{\partial \theta}$.

   (a) Suppose $f(z)$ is differentiable where $z = re^{i\theta}$. Show that $\dfrac{\partial f}{\partial r} = \dfrac{1}{r}\left[z\dfrac{\partial f}{\partial z} + \overline{z}\dfrac{\partial f}{\partial \overline{z}}\right]$ and $\dfrac{\partial f}{\partial \theta} = i\left[z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}\right]$. [Hint: Note that $\overline{z} = re^{-i\theta}$ and then use the chain rule.]

   (b) Find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ for $f(z) = z^2$ and for $f(z) = z^3 \overline{z}^3$. Do these agree with the expected expressions for $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ when f is written in terms of r and θ?
