E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 11, due Fri Dec 1st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I.5: Mild justifications are required for these problems. Answers will be graded mostly on correctness.

1. Evaluate the following real integrals, making sure to explain all steps (e.g., introducing contours, calculating residues, defining branch cuts, bounding integrands, etc.).

   (a) $\int_0^{2\pi} \frac{1}{4-\cos\theta}\,d\theta$.

   (b) $\int_0^{2\pi} \frac{1}{20\sin\theta+22}\,d\theta$.

   (c) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^4}\,dx$.

   (d) $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\,dx$.

   (e) $\int_{-\infty}^{\infty} \frac{\sin 3x}{x(x^2+1)}\,dx$.

   (f) $\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+4}\,dx$.

   (g) $\int_0^{\infty} \frac{x^\alpha}{x^2+1}\,dx$ for $0<\alpha<1$.

   (h) $\int_{-\infty}^{\infty} \frac{x^2\cos 2x}{(x^2+1)^2}\,dx$.

   (i) $\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx$.

   (j) $\int_0^{\infty} \frac{(\ln x)^2}{x^2+1}\,dx$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

2. Recall the definition of the gamma function $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\,dt$, which converges for $\text{Re}(\alpha)>0$. The goal of this problem is to prove the reflection identity $\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin(\pi\alpha)}$ for $0<\text{Re}(\alpha)<1$.

   (a) If $0<\alpha<1$, show that $\Gamma(\alpha)\Gamma(1-\alpha)=\int_0^\infty\int_0^\infty (s/t)^\alpha e^{-(s+t)}\,s^{-1}dt\,ds$.

   (b) If $0<\alpha<1$, show that $\int_0^\infty\int_0^\infty (s/t)^\alpha e^{-(s+t)}\,s^{-1}dt\,ds=\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx$. [Hint: Make a change of variables $r=s+t$ and $x=s/t$ and verify that $x^{-1}(1+x)^{-1}\,dr\,dx=s^{-1}\,dt\,ds$.]

   (c) If $0<\alpha<1$, show that $\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx=\frac{\pi}{\sin(\pi\alpha)}$. [Hint: Use the keyhole contour.]

   (d) Show that $\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin(\pi\alpha)}$ for all $\alpha\in\mathbb{C}$ with $0<\text{Re}(\alpha)<1$.

   (e) Compute $\Gamma(1/2)$ and use the result to evaluate the Gaussian integral $\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$.

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3. The goal of this problem is to evaluate the Fresnel integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$. Let $R>0$ and let $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3$ where γ_1 is the line segment from 0 to R, γ_2 is the counterclockwise circular arc of $|z|=R$ from R to $Re^{i\pi/4}$, and γ_3 is the line segment from $Re^{i\pi/4}$ to 0.

   (a) Show that $\int_\gamma e^{-z^2}\,dz=0$.

   (b) Show that $\int_{\gamma_1} e^{-z^2}\,dz\to\frac{1}{2}\sqrt{\pi}$ as $R\to\infty$. [Hint: Use 2(e).]

   (c) Show that $\int_{\gamma_2} e^{-z^2}\,dz\to 0$ as $R\to\infty$. [Hint: Use the inequality $\cos 2t\geq 1-4t/\pi$ for $0\leq t\leq\pi/4$.]

   (d) Prove that $\int_0^\infty \sin(x^2)\,dx=\int_0^\infty \cos(x^2)\,dx=\frac{\sqrt{2\pi}}{4}$.

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4. The goal of this problem is to give another another another another proof of the fundamental theorem of algebra. Let $p(z)=a_dz^d+\cdots+a_1z+a_0$ be a polynomial with $a_d\neq 0$.

   (a) Let γ_R be the counterclockwise circle $|z|=R$. Show that $\lim_{R\to\infty}\int_{\gamma_R}\frac{p'(z)}{p(z)}\,dz$ is not zero, and calculate the exact value. [Hint: Show that $\frac{p'(Re^{it})}{p(Re^{it})}Re^{it}-d$ is $O(R^{-1})$.]

   (b) Show that $p(z)$ has d zeroes (counting multiplicities) in $\mathbb{C}$.
