E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 10, due Sun Nov 20th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the requested terms in the Laurent expansion for each function $f(z)$ on the given region:
   (a) The terms from degree -5 to 5 of $f(z) = 1/(z + z^2)$ for $0 < |z| < 1$.
   (b) The terms from degree -5 to 5 of $f(z) = 1/(z + z^2)$ for $|z| > 1$.
   (c) The terms from degree -3 to 3 of $f(z) = 1/(z + z^2)$ for $0 < |z + 1| < 1$.
   (d) The terms from degree -4 to 4 of $f(z) = 1/(z + z^2)$ for $|z + 1| > 1$.
   (e) The terms from degree -3 to 3 of $f(z) = \dfrac{1}{e^z - 1}$ for $0 < |z| < 2\pi$.
   (f) The terms from degree -3 to 3 of $f(z) = \dfrac{e^{1/z}}{1 - z^2}$ for $|z| > 1$.
   (g) The terms from degree -2 to 2 of $f(z) = \dfrac{e^{1/z}}{1 - z^2}$ for $0 < |z| < 1$. [The terms will be infinite sums. You can evaluate them or leave them unsimplified.]

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2. For each function $f(z)$, identify (i) all zeroes of f and their orders, (ii) all removable singularities, (iii) all poles and their orders, and (iv) all essential singularities:
   (a) $f(z) = e^z - 1$.
   (b) $f(z) = \dfrac{1}{e^z - 1}$.
   (c) $f(z) = \dfrac{e^z - 2}{e^z - 1}$.
   (d) $f(z) = \dfrac{\sin(z^2)}{z^3}$.
   (e) $f(z) = (z^2 - 1)\sin(\pi/z)$.
   (f) $f(z) = \dfrac{e^{\pi z}\sin(\pi z)}{z(z^2 - 1)^3}$.
   (g) $f(z) = \dfrac{1}{\sin z} - \dfrac{1}{z}$. [Classify the singularities only.]
   (h) $f(z) = \dfrac{z^n}{1 - z^n}$, n a positive integer.

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3. Calculate the residue of each function $f(z)$ at the given point:
   (a) $f(z) = e^z/\sin(z)$ at $z = 0$.
   (b) $f(z) = e^z/\sin(z)$ at $z = \pi$.
   (c) $f(z) = e^z/(z + 1)^2$ at $z = -1$.
   (d) $f(z) = e^z/\sin(z^2)$ at $z = 0$.
   (e) $f(z) = \dfrac{1}{z^3(z + 1)^4}$ at $z = 0$.
   (f) $f(z) = \dfrac{1}{z^3(z + 1)^4}$ at $z = -1$.
   (g) $f(z) = e^{1/z}$ at $z = 0$.

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4. The goal of this problem is to describe how to compute a Laurent expansion for $\csc(z)$ on the annulus $\pi < |z| < 2\pi$ starting from the Laurent expansion $\csc(z) = z^{-1} + \dfrac{1}{6}z + \dfrac{7}{360}z^3 + \dfrac{31}{15120}z^5 + \cdots$.
   (a) Find the terms in the Laurent expansion of $\csc(z)$ centered at $z = -\pi$ and $z = \pi$ up to degree 5. [Hint: What is $\csc(z \pm \pi)$?]
   (b) Verify that $\csc(z) + \dfrac{2z}{z^2 - \pi^2} = \csc(z) + \dfrac{1}{z - \pi} + \dfrac{1}{z + \pi}$ has removable singularities at $z = -\pi$ and $z = \pi$. Deduce that it has a Laurent expansion centered at $z = 0$ with radius of convergence 2π, and find the terms up to degree 5.
   (c) Explain why $\csc(z)$ has a Laurent expansion on the annulus $\pi < |z| < 2\pi$ and compute its terms from degree -5 to degree 5.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Suppose that $f(z)$ is bounded and holomorphic on $\mathbb{C}\backslash\{0\}$. Show that $f(z)$ is constant.

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6. Suppose $f(z)$ is meromorphic on a region R and that f has an isolated singularity at $z = z_0$.

   (a) If $\lim_{z\to z_0}(z - z_0)f(z) = 0$, show that f has a removable singularity at z_0.

   (b) If f has a zero of order k at z_0, show $f'(z)/f(z)$ has a simple pole at z_0 with residue k.

   (c) If f has a pole of order k at z_0, show $f'(z)/f(z)$ has a simple pole at z_0 with residue $-k$.

   (d) [Optional] If f has a pole at z_0, show that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ take arbitrarily large positive and negative values as $z \to z_0$. [Hint: If $f(z) = re^{i\theta}(z - z_0)^{-k} + \cdots$, take $z = z_0 + t^{1/k}e^{i\theta/k}$ as $t \to 0$ in different directions.]

   (e) If f has an essential singularity at z_0, show that e^f also has an essential singularity at z_0.

   (f) If f has a pole at z_0, show that e^f has an essential singularity at z_0. [Hint: Use (c).]

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7. The goal of this problem is to give another another another proof of the fundamental theorem of algebra, due to Schep. Let $p(z)$ be a nonconstant polynomial that has no roots.

   (a) Show that $\dfrac{1}{zp(z)}$ has a single simple pole at $z = 0$, and calculate its residue there and show the residue is nonzero.

   (b) If γ_r is the counterclockwise circle $|z| = r$, show that $\int_{\gamma_r} \dfrac{1}{zp(z)}\,dz \to 0$ as $r \to \infty$. Obtain a contradiction. [Hint: Suppose $|p(z)|$ has minimum M_r on γ_r. As shown in class, $M_r \to \infty$ as $r \to \infty$.]
