E. Dummit's Math 4555 ~ Complex Analysis, Fall 2022 ~ Homework 1, due Wed Sep 14th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Express the following complex numbers in rectangular $a + bi$ form:

   (a) $(3+i)-(4-2i)$. (b) $(8-6i)(2+3i)$. (c) $(4+3i)/(5-i)$. (d) $e^{i\pi/4}$. (e) $4e^{7i\pi/6}$. (f) $3e^{-3i\pi/4}$. (g) $e^{2022i\pi/3}$. (h) $\pi e^{\pi i/e}$. (i) i^{2022}. (j) $(1+i)^{12}$.

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2. Express the following complex numbers in polar $re^{i\theta}$ form:

   (a) $3i$. (b) $-2-2i\sqrt{3}$. (c) $-1+i$. (d) $3+4i$. (e) $\sqrt{-2}$. (f) $(e^{i\pi/6})^{2022}$. (g) $(-1+i)^{2022}$. (h) $\pi + ei$.

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3. Find all $z \in \mathbb{C}$ satisfying the following equations:

   (a) $z^2 + 2z + 2 = 0$. (b) $z^2 - z + 6 = 0$. (c) $z^2 = 3 + 4i$. (d) $z^2 - (2+2i)z + 10i = 0$. (e) $z^3 = -1$. (f) $z^8 = 1$. (g) $z^4 = 3$. (h) $e^z = 1$. (i) $e^z = 1 - i\sqrt{3}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. The real numbers are an example of an <u>ordered field</u>, which is a field containing a subset P (the "positive" elements) such that (i) P is closed under addition, (ii) P is closed under multiplication, and (iii) every nonzero element of the field is either in P or its additive inverse is in P but not both. Prove that $\mathbb{C}$ is not an ordered field for any possible choice of P. [Hint: Consider i.]

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5. Suppose z and w are complex numbers. The goal of this problem is to study some properties of the rational function $f(z) = \dfrac{z-w}{1-z\overline{w}}$, which when $|w| < 1$ is called a <u>Möbius transformation</u>.

   (a) Show that $f(z) = \dfrac{z-w}{1-z\overline{w}}$ is one-to-one and that its inverse is $f^{-1}(z) = \dfrac{z+w}{1+z\overline{w}}$.

   (b) If $\overline{z} \cdot w \neq 1$, show that $1 - \left|\dfrac{z-w}{1-z\overline{w}}\right|^2 = \dfrac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$.

   (c) If $|w| < 1$, show that f is a bijection that maps the interior of the unit disc to itself (i.e., that if $|z| < 1$ then $|f(z)| < 1$) and also maps the boundary of the unit disc to itself (i.e., that if $|z| = 1$ then $|f(z)| = 1$).

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6. The goal of this problem is to illustrate some uses of complex exponentials for trigonometry. You may assume for the purposes of this problem that indefinite integrals involving complex numbers behave the expected way (we will later prove this).

   (a) If x is real, show that $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.

   (b) Compute $\int \cos^4 x\,dx$ and $\displaystyle\int \sin^4 x\,dx$. [Hint: Use (a).]

   (c) Compute $\displaystyle\int e^{ax} \cos bx\,dx$ and $\displaystyle\int e^{ax} \sin bx\,dx$. [Hint: Take real and imaginary parts of $\displaystyle\int e^{(a+bi)x}\,dx$.]

   (d) Prove that $1 + e^{ix} + e^{2ix} + \cdots + e^{inx} = \dfrac{e^{(n+1)ix} - 1}{e^{ix} - 1} = e^{(n/2)ix} \dfrac{\sin[\frac{n+1}{2}x]}{\sin(x/2)}$ for any positive integer n and any $0 < x < 2\pi$.

   (e) Prove that $1 + \cos x + \cos 2x + \cdots + \cos nx = \dfrac{\sin[\frac{n+1}{2}x]\cos[\frac{n}{2}x]}{\sin(x/2)}$ and $\sin x + \sin 2x + \cdots + \sin nx = \dfrac{\sin[\frac{n+1}{2}x]\sin[\frac{n}{2}x]}{\sin(x/2)}$ for any positive integer n and any $0 < x < 2\pi$.

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7. The goal of this problem is to illustrate one of the original historical applications of the complex numbers: that of solving the cubic equation.

   (a) Suppose that $z^3 + az^2 + bz + c = 0$. Show that $t = z + a/3$ has $t^3 + pt + q = 0$ where $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$. Thus, it suffices to solve cubics of the form $t^3 + pt + q = 0$.

   (b) Suppose that $t^3 + pt + q = 0$. Define new variables x and y such that $x + y = t$ and $3xy = -p$. Show that $x^3 + y^3 = -q$ and then solve for x^3 and y^3.

   (c) Conclude that the solutions to the cubic $t^3 + pt + q = 0$ are the three numbers of the form $t = A + B$, with $A = \sqrt[3]{-\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$ and $B = \sqrt[3]{-\dfrac{q}{2} - \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$, where the cube roots are selected so that $AB = -p/3$. These formulas are known as Cardano's formulas.

   (d) Verify that the cubic $f(t) = t^3 - 15t - 4$ has three real roots and that they are 4 and $-2 \pm \sqrt{3}$.

   (e) Use Cardano's formulas to find the roots of $f(t) = t^3 - 15t - 4$, and then show that they do simplify to yield the same answers in (d) using the calculation $(2+i)^3 = 2 + 11i$.

   - Remark: The calculation in (e) was performed by Bombelli in 1572. This rather perplexing appearance of square roots of negative numbers in the formulas for real solutions to cubic equations was the original impetus that led to the development and acceptance of complex numbers in mathematics (although unsurprisingly, it did take a while!).
