E. Dummit's Math 4527 ~ Number Theory 2, Fall 2022 ~ Homework 5, due Fri Oct 14th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Determine, with brief reasons, whether each subset S is an ideal of the given ring R:

   (a) $R = F[x]$, $S =$ the set of polynomials whose coefficient of x is zero.
   (b) $R = \mathbb{Z}/18\mathbb{Z}$, $S = \{0, 3, 6, 9, 12, 15\}$, the set of multiples of 3 in R.
   (c) $R = \mathbb{Z}/15\mathbb{Z}$, $S = \{0, 4, 8, 12\}$, the set of multiples of 4 in R.
   (d) $R = \mathbb{Z} \times \mathbb{Z}$, $S = \{(a, a) : a \in \mathbb{Z}\}$.
   (e) $R = \mathbb{Z} \times \mathbb{Z}$, $S = \{(0, a) : a \in \mathbb{Z}\}$.
   (f) $R = F[x]$, $S = F[x^2]$, the polynomials in which only even powers of x appear.
   (g) $R = F[x]$, $S =$ the set of polynomials whose coefficients sum to zero.

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2. Let $R = \mathbb{Z}[\sqrt{7}]$ and consider the ideals $I = (3)$ and $J = (3, 1 + \sqrt{7})$.

   (a) Show that R/I contains exactly 9 residue classes. [Hint: They are $p + q\sqrt{7} + I$ for $p, q \in \{0, 1, 2\}$. Explain why.]
   (b) Write down the multiplication table for R/I, and identify which elements are units and which elements are zero divisors. Is I a prime ideal? A maximal ideal?
   (c) Show that R/J contains exactly 3 residue classes and identify them. Is J a prime ideal? A maximal ideal?

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Let R be a commutative ring with 1.

   (a) Show with an explicit example that the union of a collection of ideals of R is not necessarily an ideal of R.
   (b) If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an increasing chain of ideals of R, show that the union $\bigcup_{i=1}^{\infty} I_i$ is an ideal of R.

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4. Let R be a commutative ring with 1 and let I and J be ideals of R.

   (a) Show that $I + J = \{a + b : a \in I, b \in J\}$, the set of all sums of elements of I and J, is an ideal of R.
   (b) Show that $I + J$ is the smallest ideal of R that contains both I and J. Deduce that if $I = (a_1, \dots, a_n)$ and $J = (b_1, \dots, b_m)$ then $I + J = (a_1, \dots, a_n, b_1, \dots, b_m)$.
   (c) Let a and b be positive integers with gcd d. Show that $(a) + (b) = (d)$ in $\mathbb{Z}$.
   (d) Show that $IJ = \{a_1 b_1 + \cdots + a_n b_n, : a_i \in I, b_i \in J\}$, the set of finite sums of products of an element of I with an element of J, is an ideal of R.
   (e) If $I = (a_1, \dots, a_n)$ and $J = (b_1, \dots, b_m)$, show that $IJ = (a_1 b_1, a_1 b_2, \dots, a_n b_1, a_1 b_2, \dots, a_n b_m)$.
   (f) Show that IJ is an ideal contained in $I \cap J$, and give an example where $IJ \neq I \cap J$.
   (g) If $I + J = R$, show that $IJ = I \cap J$. [Hint: There exist $x \in I$ and $y \in J$ with $x + y = 1$.]

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5. Let R be a commutative ring with 1 and define the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$. Prove the binomial theorem in R: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for any $x, y \in R$ and any $n > 0$.

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6. Let R be a commutative ring with 1. We say $x \in R$ is <u>nilpotent</u> if $x^n = 0$ for some positive integer n.

   (a) Find the nilpotent elements of $\mathbb{Z}/12\mathbb{Z}$.

   (b) If m is a positive integer, show that a is nilpotent in $\mathbb{Z}/m\mathbb{Z}$ if and only if every prime divisor of m also divides a.

   (c) Show that the set of nilpotent elements of R forms an ideal of R; this ideal is called the <u>nilradical</u> of R. [Hint: Use the binomial theorem to establish closure under subtraction.]

   (d) If x is nilpotent, show that $1+x$ is a unit. [Hint: What is $(1+x)(1-x+x^2-x^3+\cdots)$?]

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7. [Challenge] Let R be a commutative ring with 1 and let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial in $R[x]$. The goal of this problem is to characterize the units and nilpotent elements in $R[x]$.

   (a) If $a_0, a_1, \ldots, a_n$ are all nilpotent in R, show that $p(x)$ is nilpotent in $R[x]$.

   (b) If $p(x)$ is nilpotent in $R[x]$, show that $a_0, a_1, \ldots, a_n$ are all nilpotent in R. [Hint: Show that a_n must be nilpotent and then induct.]

   (c) If a_0 is a unit and $a_1, a_2, \ldots, a_n$ are nilpotent in R, show that $p(x)$ is a unit in $R[x]$.

   (d) If $p(x)$ is a unit in $R[x]$, show that a_0 is a unit and $a_1, a_2, \ldots, a_n$ are nilpotent in R. [Hint: If the inverse is $b_0 + b_1 x + \cdots + b_m x^m$, show that b_0 is a unit and also that $a_n^{k+1} b_{m-k} = 0$ for each $k \geq 0$, and deduce that a_n is nilpotent.]
