E. Dummit's Math 4527 ~ Number Theory 2, Fall 2022 ~ Homework 11, due Fri Dec 9th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Calculate the number of Dirichlet characters (a) modulo 5, and (b) modulo 8, and compute their values explicitly.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

2. The goal of this problem is to evaluate some Dirichlet L-series at 1.
   (a) Let χ_4 be the nontrivial Dirichlet character mod 4. Show $L(1,\chi_4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.
   (b) Let $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| < 1$. Show that $F'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$ and deduce that $L(1,\chi_4) = F(1) = \int_0^1 \frac{1}{1+x^2}\,dx = \pi/4$. [Hint: Since the series for F converges absolutely, it can be differentiated term by term.]
   (c) Let χ_3 be the nontrivial Dirichlet character modulo 3. Show that $L(1,\chi_3) = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}$.
   (d) Let $G(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+1)(3n+2)}$ for $|x| < 1$. Show that $G(1) = \int_0^1 \int_0^y \frac{1}{1-x^3}\,dx\,dy$ and use this to compute the value of $L(1,\chi_3)$. [Hint: Note that $G''(x) = (1-x^3)^{-1}$ for $|x| < 1$.]

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3. The Carmichael Λ-function $\Lambda(n)$ is defined to be $\ln(p)$ if $n = p^d$ is a prime power and 0 otherwise. It is frequently used in proofs of the prime number theorem.
   (a) Show that $\sum_{d|n} \Lambda(d) = \ln n$.
   (b) Show that the Dirichlet series for Λ is $D_\Lambda(s) = -\zeta'(s)/\zeta(s)$ for $\text{Re}(s) > 1$.
   (c) Show that $D_\Lambda(s) = \sum_{p \text{ prime}} \frac{\ln p}{p^s - 1}$ for $\text{Re}(s) > 1$. [Hint: Use (b) and logarithmic differentiation.]

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4. The goal of this problem is to show that the set of primes whose leading digit is 1 in base 10 has undefined natural density using the following weak form of the prime number theorem: for sufficiently large n, the number of primes $\pi(n)$ less than n is between $0.99\frac{x}{\ln x}$ and $1.01\frac{x}{\ln x}$.
   (a) Show that for sufficiently large k, there are at most $\frac{1.01}{k \ln 10} 10^k$ primes below 10^k and at least $\frac{1.97}{k \ln 10} 10^k$ primes below $2 \cdot 10^k$.
   (b) Show that for sufficiently large k, a proportion at least $0.96/2.02$ of primes below $2 \cdot 10^k$ have leading digit 1.
   (c) Show that for sufficiently large k, a proportion at most $2.02/9.9$ of primes below $10 \cdot 10^k$ have leading digit 1.
   (d) Deduce that the natural density of the set of primes of leading digit 1 is undefined.

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5. Let G be a finite abelian group with dual group $\hat{G}$, and recall the inner products $\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$ on functions $f : G \to \mathbb{C}$ and $\left\langle \hat{f}_1, \hat{f}_2 \right\rangle_{\hat{G}} = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}_1(\chi)\overline{\hat{f}_2(\chi)}$ on functions $\hat{f} : \hat{G} \to \mathbb{C}$. Also recall the Fourier transform of a function $f : G \to \mathbb{C}$ is the function $\hat{f} : \hat{G} \to \mathbb{C}$ with $\hat{f}(\chi) = \langle f, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi(g)}$, and the Fourier inversion formula $f(g) = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(g)$ for each $g \in G$.

   (a) Prove Plancherel's theorem: $\frac{1}{|G|} \langle f_1, f_2 \rangle_G = \left\langle \hat{f}_1, \hat{f}_2 \right\rangle_{\hat{G}}$ for any functions $f_1, f_2 : G \to \mathbb{C}$. [Hint: Write $\hat{f}_1(\chi)$ as a sum over $g \in G$ and $\hat{f}_2(\chi)$ as a sum over $h \in G$, then use the fact that $\sum_{\chi \in \hat{G}} \overline{\chi(g)}\chi(h)$ is either $|G|$ or 0 according to whether $g = h$ or not.]

   (b) Deduce Parseval's theorem: $\frac{1}{|G|} \sum_{g \in G} |f(g)|^2 = \sum_{\chi \in \hat{G}} \left|\hat{f}(\chi)\right|^2$.

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6. The goal of this problem is to give another estimate for the absolute value of the Dedekind zeta function. Let $R = \mathcal{O}_{\sqrt{D}}$ be a quadratic integer ring with associated Dedekind zeta function $\zeta_K(s) = \sum_{I \subseteq R} N(I)^{-s}$, and let $f(n)$ be the number of ideals of R of norm n.

   (a) Show that $\zeta_K(s)$ is the Dirichlet series $D_f(s)$ for $f(n)$.

   (b) Show that $f(n) \leq d(n)$ where d is the divisor-counting function. [Hint: $f(n)$ is multiplicative.]

   (c) Show that $|\zeta_K(s)| \leq \zeta(\mathrm{Re}(s))^2$.

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7. [Challenge] Let p be a prime and let χ be the Legendre symbol modulo p. The goal of this problem is to evaluate $L(1, \chi)$ explicitly and then to prove a formula for the class number in terms of the number of quadratic residues and nonresidues on the interval $[1, (p-1)/2]$ when $p \equiv 3 \pmod 4$. Recall the Gauss sum $g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta^n$ where $\zeta = e^{2\pi i/p}$, and the general Gauss sum $g_k(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta^{kn}$.

   (a) Show that $-\log(1 - \zeta^n) = \sum_{k=1}^{\infty} \frac{\zeta^{nk}}{k}$. (Note that this series only converges conditionally.)

   (b) Let $S = \sum_{n=1}^{p-1} \chi(n) \cdot [-\log(1 - \zeta^n)]$. Prove that $S = g(\chi)L(1, \chi)$. [Hint: Use (a), switch summation order, and use the Gauss sum identity $g_k(\chi) = \chi(k)^{-1}g(\chi)$.]

   (c) Define $P = \frac{\prod_{n \in NR}(1 - \zeta^n)}{\prod_{n \in QR}(1 - \zeta^n)}$ where NR is the set of quadratic nonresidues modulo p and QR is the set of quadratic residues modulo p. Show that $P = \exp(g(\chi)L(1, \chi))$.

   (d) Find the value of $L(1, \chi)$ for the Legendre symbol modulo 3. [Hint: The result of (c) is easier to calculate with, unless you like complex logarithms.]

   (e) Show that if $p \equiv 3 \pmod 4$, so that $\chi(-1) = -1$, then $S = -\frac{i\pi}{p} \sum_{n=1}^{p-1} \chi(n) \cdot n$ where S is as defined in (b). [Hint: In (b), interchange n with $-n$ and add the two sums together.]

   (f) Show that when $p \equiv 3 \pmod 4$ and $p > 3$ we have $h(-p) = -\frac{1}{p} \sum_{n=1}^{p-1} \chi(n) \cdot n$. [Hint: Use the Gauss sum evaluation $g(\chi) = i\sqrt{p}$ and the analytic class number formula.]

   (g) Show that when $p \equiv 3 \pmod 4$ and $p > 3$ we have $h(-p) = \frac{1}{2 - \chi(2)} \sum_{n=1}^{(p-1)/2} \chi(n)$. [Hint: Decompose $\sum_{n=1}^{p-1} \chi(n) \cdot n$ into two ranges in two different ways: one into even and odd, and another into $[1, (p-1)/2]$ and $p - [1, (p-1)/2]$.]

   (h) Deduce that when $p \equiv 3 \pmod 4$, the class number of $\mathcal{O}_{\sqrt{-p}}$ is equal to $\frac{1}{2 - \chi(2)}$ times the number of quadratic residues in $[1, (p-1)/2]$ minus the number of quadratic nonresidues on that interval, so in particular there are always more quadratic residues than quadratic nonresidues. Also deduce in particular that this class number is always odd.

   (i) Find the class numbers of $\mathcal{O}_{\sqrt{-7}}$, $\mathcal{O}_{\sqrt{-11}}$, $\mathcal{O}_{\sqrt{-19}}$, and $\mathcal{O}_{\sqrt{-31}}$. [If you're still here at this point, for convenience $\chi(2) = 1$ when $p \equiv 7 \pmod 8$ and $\chi(2) = -1$ when $p \equiv 3 \pmod 8$.]
