E. Dummit's Math 4527 $\sim$ Number Theory 2, Fall 2022 $\sim$ Homework 1, due Fri Sep 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find all solutions in integers (if any) to the following linear Diophantine equations:
   (a) $22a + 17b = 19$.
   (b) $42a + 27b = 39$.
   (c) $3a + 7b + 16c = 8$.

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2. Find all right triangles having one leg of length 40, and whose other two side lengths are integers.

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3. Byzantine Basketball is like regular basketball except that foul shots are worth a points instead of two points and field shots are worth b points instead of three points. Moreover, in Byzantine Basketball there are exactly 35 scores that never occur in a game, one of which is 58. What are a and b?

   Remark: This problem was on the 1971 Putnam exam.

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4. Compute the following things:
   (a) Show that $7/13$ and $13/24$ are adjacent in the Farey sequence of level 24. What are the next three terms after them?
   (b) Find all n such that exactly 2 terms appear between $7/13$ and $13/24$ in the Farey sequence of level n.
   (c) List all the terms between $6/19$ and $5/14$ in the Farey sequence of level 19.
   (d) Find the three terms following $154/227$ in the Farey sequence of level 2022.
   (e) List all the terms between $1502/1801$ and $1492/1789$ in the Farey sequence of level 2022.
   (f) Find the least possible difference between two consecutive terms in the Farey sequence of level 2022.
   (g) Find the greatest possible difference between two consecutive terms in the Farey sequence of level 2022.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. The goal of this problem is to discuss the Weierstrass substitution $t = \tan(\theta/2)$.
   (a) Show that $\tan(\theta/2) = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$. [Hint: Draw the triangle with vertices $(-1, 0)$, $(0, 0)$, and $(\cos\theta, \sin\theta)$ inside the unit circle.]
   (b) Let $t = \tan(\theta/2)$. Show that $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, and $d\theta = \dfrac{2dt}{1+t^2}$. [Hint: Solve the identity in (a) for $\cos\theta$ and $\sin\theta$.]
   (c) Compute the indefinite integral $\displaystyle\int \frac{1}{5+3\cos\theta}\,d\theta$. [Yes, you can just ask any computer algebra system to do this for you, but then you wouldn't learn how to do it yourself.]

   - Remark: The method in (c) generalizes to allow evaluation of any integral that is a rational function of $\sin\theta$ and $\cos\theta$.

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6. If n is any positive integer greater than 2, prove that there exists at least one right triangle with one side of length n and whose other sides have integer lengths.

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7. Prove that the area of any right triangle with integer sides is always divisible by 6, but not necessarily any integer greater than 6.

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8. The goal of this problem is to find all solutions to the Diophantine equation $x^2 + y^2 = 2z^2$ in various ways.

   (a) Suppose $x^2 + y^2 = 2z^2$ where x, y, z are integers. Show that x and y must have the same parity, and then that if $a = (x + y)/2$ and $b = (x - y)/2$, there must exist integers k, s, t such that $a = 2kst$ and $b = k(s^2 - t^2)$ or vice versa.

   (b) Suppose $x^2 + y^2 = 2z^2$ where x, y, z are integers. Show that $1 + i$ must divide both $x + iy$ and $x - iy$ in the Gaussian integers $\mathbb{Z}[i]$, and then that if $p + iq = (x + iy)/(1 + i)$, there exist integers k, s, t such that $p = k(s^2 - t^2)$ and $q = 2kst$.

   (c) Suppose $x^2 + y^2 = 2z^2$ where x, y, z are integers and $z \neq 0$. Show that the line through $(x/z, y/z)$ and $(-1, 1)$ has rational slope. Also, if ℓ is the line with rational slope t/s through the point $(-1, 1)$, find the intersection point of ℓ with the circle $(x/z)^2 + (y/z)^2 = 2$.

   (d) Find all solutions to the Diophantine equation $x^2 + y^2 = 2z^2$.

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9. For each Farey fraction a/b, define $\mathcal{C}(a/b)$ to be the circle in the upper-half of the Cartesian plane of radius $r_{a/b} = 1/(2b^2)$ that is tangent to the x-axis at the point $(a/b, 0)$. These circles are called the Ford circles.

   (a) If a/b and c/d are adjacent entries in some Farey sequence, prove that the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are tangent.

   (b) If a/b and c/d are not adjacent in any Farey sequence, prove that the interiors of the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are disjoint.

   (c) Draw (you may use a computer) the 11 Ford circles corresponding to the Farey sequence of level 5.

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10. [Challenge] Let a, b, c be pairwise relatively prime positive integers. Show that $2abc - ab - bc - ca$ is the largest integer that cannot be expressed in the form $xbc + yca + zab$ for nonnegative integers x, y, z. [Hint: Any integer is congruent modulo a to precisely one of 0, bc, $2bc$, ... , $(a - 1)bc$.]

    - Remark: This is a Frobenius coin problem for the integers ab, bc, ca and was problem 3 from the 1983 IMO.
