1. Decide whether each residue class has a multiplicative inverse modulo m. If so, find it, and if not, explain why not:

   (a) $\overline{10}$ mod 25.
   (b) $\overline{11}$ mod 25.
   (c) $\overline{12}$ mod 25.
   (d) $\overline{30}$ mod 42.
   (e) $\overline{31}$ mod 42.
   (f) $\overline{32}$ mod 42.

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2. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) irreflexive, (vi) an equivalence relation, (vii) a partial ordering, and (viii) a total ordering.

   (a) $R = \{(1,1),\ (2,1),\ (2,2)\}$ on the set $\{1,2\}$.
   (b) $R = \{(1,2),\ (2,1)\}$ on the set $\{1,2\}$.
   (c) $R = \{(1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,3),\ (3,4),\ (4,3),\ (4,4)\}$ on the set $\{1,2,3,4\}$.
   (d) The divisibility relation on the set $\{2,-3,4,-5,6\}$.
   (e) The divisibility relation on the set $\{2,-4,-12,36\}$.
   (f) The relation R on $\mathbb{Z}$ with $a\ R\ b$ precisely when $|a| \equiv |b|$ modulo 5.
   (g) The relation R on $\mathbb{R}$ with $a\ R\ b$ precisely when $ab > 0$.

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3. For each of the following functions $f : A \to B$, determine whether (i) f is one-to-one, (ii) f is onto, (iii) f is a bijection.

   (a) $f(x) = 2x$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
   (b) $f(n) = 2n$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.
   (c) $f(x) = \dfrac{x}{x-1}$ from $A = \mathbb{R}\backslash\{1\}$ to $B = \mathbb{R}$.
   (d) $f(x) = x^3$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.
   (e) $f = \{(1,2),(2,3),(3,4),(4,1)\}$ from $\{1,2,3,4\}$ to itself.
   (f) $f = \{(1,3),(2,4),(3,1),(4,4)\}$ from $\{1,2,3,4\}$ to itself.

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4. Identify the ordered pairs in the equivalence relation that corresponds to the partition $\{1,2,4\},\{3,5\},\{6\}$ of $\{1,2,3,4,5,6\}$.

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5. Show $R = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : |x| = |y|\}$ is an equivalence relation on $\mathbb{Z}$ and list the equivalence classes of 0, 1, 2, -2, 4.

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6. Find all minimal, maximal, largest, and smallest elements of the set $\{-2,-1,1,2,3,6\}$ under the divisibility ordering, and on the ordering $a\,R\,b$ when $a = b$ or $a + 2 < b$.

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7. Suppose $f : A \to B$ is a function.

   (a) If f is one-to-one, show that there is a bijection between A and $\mathrm{im}(f)$. Deduce that $\#A = \#\mathrm{im}(f)$.

   (b) If A and B are both finite and $\#A = \#B$, show that f is one-to-one implies that f is onto.

   (c) Show that (b) is false for infinite sets by giving a function $f : \mathbb{R} \to \mathbb{R}$ where f is one-to-one but not onto.

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8. A real number is <u>algebraic</u> if it is a root of a nonzero polynomial $p(x)$ with integer coefficients, and it is <u>transcendental</u> if it is not the root of any such polynomial.

   (a) Let S_n be the set of all roots of nonzero polynomials of degree at most n whose coefficients are integers and at most n in absolute value. Show that S_n is finite and that the set of algebraic numbers is $\cup_{n\geq 1} S_n$.

   (b) Show that the set of algebraic numbers is countable. Deduce there are uncountably many transcendental numbers.

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9. Let A, B, C be sets, R be a relation, f, g be functions, and n be a positive integer. Prove the following:

   (a) Prove that the sum of any six consecutive integers is congruent to 3 modulo 6.

   (b) Show that the only equivalence relation R on A that is a function from A to A is the identity relation.

   (c) Prove that $5^n + 6^n \equiv 0 \pmod{11}$ if and only if n is odd.

   (d) Prove that if A is countable and B is uncountable, then $B \backslash A$ is uncountable.

   (e) Suppose A, B are fixed sets and let T be the collection of all sets D with $D \subseteq A$ and $D \subseteq B$. Show that $A \cap B$ is the largest element of T.

   (f) Prove that the set $\mathbb{Q} \times \mathbb{Z}$ is countable and that the set $\mathbb{R} \times \mathbb{Z}$ is uncountable.

   (g) Prove that $7^n + 5$ is divisible by 6 for all positive integers n.

   (h) If $f : A \to B$ is a function, show that $R = \{(a, a') \in A \times A \,:\, f(a) = f(a')\}$ is an equivalence relation on A.

   (i) Suppose $f : B \to C$ is one-to-one. If $g, h : A \to B$ have $f \circ g = f \circ h$, show that $g = h$.

   (j) If n is any positive integer, prove that $n - 1$ is invertible modulo n and its multiplicative inverse is itself.

   (k) Prove that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$, the set of rational numbers strictly between 0 and 1.

   (l) Prove that if $f : A \to B$ is one-to-one and $S \subseteq A$, then $f^{-1}(f(S)) = S$.

   (m) Prove that if $f : A \to B$ is onto and $T \subseteq B$, then $f(f^{-1}(T)) = T$.

   (n) Suppose $f : A \to B$ is a bijection. Show that $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ given by $\tilde{f}(S) = \{f(s) \,:\, s \in S\}$ is also a bijection.

   (o) If $R, S : A \to B$ are relations, prove that $R^{-1} \cap S^{-1} = (R \cap S)^{-1}$.

   (p) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Show that $a(b + c) \equiv b(a + d) \pmod{n}$.
