1. (a) Not equivalent (b) Not equivalent (c) Equivalent (d) Not equivalent (e) Not equivalent (f) Equivalent

2. (a) True (b) False (c) False (d) True (e) False (f) True (g) False (h) True (i) True (j) False

3. (a) $\{1\}$ (b) $\{1, 2, 3, 5, 7, 9\}$ (c) $\{4, 6, 8\}$ (d) $\{3, 5, 6, 7, 9\}$ (e) $\emptyset = \{\}$ (f) $\{(1, 2), (3, 2), (5, 2), (7, 2), (9, 2)\}$
(g) $\{(1, 2)\}$ (h) $\{(1, 1), (1, 3), (3, 1), (3, 3)\}$

4. (a) $\exists x \exists y \forall z,\ x + y + z \le 5$
(b) There exists an integer that is not a rational number.
(c) $\exists x \in A\, \exists y \in B,\ x \cdot y \notin A \cap B$.
(d) Every perfect square is even.
(e) The integer n is either not prime or $n \ge 10$.
(f) $\exists \epsilon > 0\, \forall \delta > 0,\ (|x - a| < \delta) \wedge (|x^2 - a^2| \ge \epsilon)$.
(g) There exists an $x \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$, $x \ge n$.
(h) For all positive integers a and b, $\sqrt[3]{2} \ne a/b$.

5. (a) False (b) True (c) False (d) True (e) False (f) True (g) True (h) True

6. (a) gcd 8, lcm $256 \cdot 520/8$. (b) gcd 3, lcm $921 \cdot 177/3$. (c) gcd 1, lcm $2019 \cdot 5678$. (d) gcd $2^3 3^2 5^4$, lcm $2^4 3^3 5^4 7 \cdot 11$.

7. (a) True. Note $x \in (A \cup B)\backslash A$ iff $x \in (A \cup B) \cap A^c$ iff $x \in B \cap A^c$ iff $x \in B\backslash A$.
(b) False. Counterexample: $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$. Then $A\backslash(B \cap C) = \{1, 2\}$ while $(A\backslash B) \cap (A\backslash C) = \emptyset$.
(d) False. Counterexample: $A = \{1\}$, $B = \{1, 2\}$ with $U = \{1, 2\}$. Then $(A \cap B)^c \cup B = \{1, 2\}$ while $(A^c \cap B)^c = \{1\}$.
(e) True. Note $(A\backslash B)^c = (A \cap B^c)^c = A^c \cup B$, and similarly $(B\backslash A)^c = A \cup B^c$. If $x \in A^c \cap B^c$ then $x \in A^c \cup B$ and also $x \in A \cup B^c$.

8. (a) If $3a - 9b = 2$, then a and b cannot both be integers. Proof: By contradiction, if a and b are integers, then 3 divides $3a - 9b$ but 3 does not divide 2 (impossible).
(b) If $a > 1$ and $b > 1$, then $ab \ne 1$. Proof: If $a > 1$ and $b > 1$ then multiplying $a > b$ by b yields $ab > b > 1$ so $ab > 1$. In particular $ab \ne 1$.
(c) If n is even, then $5n + 1$ is odd. Proof: If $n = 2k$ then $5n + 1 = 10k + 1 = 2(5k) + 1$ is odd by definition.
(d) If n is even then n^3 is even. Proof: If $n = 2k$ then $n^3 = 8k^3 = 2(4k^3)$ is even by definition.
(e) If n is the sum of 3 consecutive integers, then n is a multiple of 3. Proof: If $n = a + (a + 1) + (a + 2)$ then $n = 3a + 3 = 3(a + 1)$ is a multiple of 3.
(f) If n divides a or n divides b then n divides ab. Proof: If $n|a$ then $a = kn$ so $ab = (kb)n$, and if $n|b$ then $b = ln$ so $ab = (al)n$. In either case, $n|ab$.

9. There are many examples for each part. Here is one for each:
(a) Example: $a = 2$, $b = 4$, $c = 6$.
(b) Example: $p = 2$, $q = 3$, then $p + q = 5$ is prime.
(c) Example: $n = 11$, then $n^2 + n + 11 = 11 \cdot 13$ is not prime.
(d) Example: $a = 12$, $b = 11$, then $a^2 - b^2 = 144 - 121 = 23$.
(e) Example: $\sqrt{2} + (-\sqrt{2}) = 0$ is rational, but $\sqrt{2}$ and $-\sqrt{2}$ is irrational.
(f) Example: $\sqrt{4} = 2$ is rational.
(g) Example: $n = -3$, then $n \ne 3$ but $n^2 = 9$.
(h) Example: $m = 3$, $n = 2$, then $m^2 - 2n^2 = 9 - 8 = 1$.
(i) Example: $x = -1$, then there is no possible y with $y^4 = x$.
(j) Examples: $2^2 + 2^2 = 2^3$, or $5^2 + 10^2 = 5^3$.

10. Here are brief outlines of each proof:

(a) Induct on n. Base case $n = 1$ has $F_1 + F_3 = 3 = F_4$. Inductive step: if $F_1 + \cdots + F_{2n+1} = F_{2n+2}$ then $F_1 + \cdots + F_{2n+1} + F_{2n+3} = [F_1 + \cdots + F_{2n+1}] + F_{2n+3} = F_{2n+2} + F_{2n+3} = F_{2n+4}$ as required.

(b) Clearly, if $6|n$ then $2|n$ and $3|n$. For the other direction, if $2|n$ then $n = 2k$. Then if $3|2k$ we must have $3|k$ since $3 \nmid 2$ and 3 is prime. So $k = 3a$, and thus $n = 6a$, meaning $6|n$.

(c) Induct on n. Base case $n = 1$ has $1 = 2 - 1/2^0$. Inductive step: If $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$, then $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}$ as required.

(d) If $p|a \cdot a$ then $p|a$ or $p|a$ by the prime divisibility property. Since the two conclusion statements are the same, we have $p|a$.

(e) Note that $33 + 9b$ is divisible by 3 but not 9. But then a^2 is divisible by 3 by (d), which would mean $3|a$ and thus $9|a$, but this is impossible.

(f) If p is a prime with $p|k^2$ and $p|(k+1)^2$, then by (d) we have $p|k$ and $p|(k+1)$ so that $p|(k+1) - k = 1$, impossible.

(g) Induct on n. Base case $n = 1$ has $\frac{1}{1 \cdot 2} = \frac{1}{2}$. Inductive step: if $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$ then $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ as required.

(h) First, $A \subseteq B$ because if $n = 4a + 6b$ then $n = 2(2a + 3c) \in B$. Also, $B \subseteq A$ because if $n = 2c$ then we would have $n = 4(2c) + 6(-c) \in A$ via Euclidean algorithm calculation.

(i) Note $\gcd(n, n+p) = \gcd(n, p)$ by gcd properties. Then $\gcd(n, p)$ divides p so is either 1 or p, and it is equal to p if and only if $p|n$ (by definition of gcd).

(j) Induct on n. Base case $n = 1$ has $a_1 = 3^1 - 2$. Inductive step: if $a_n = 3^n - 2$ then $a_{n+1} = 3(3^n - 2) + 4 = 3^{n+1} - 2$ as claimed.

(k) If $n \in C$, then $n = 6c$ for some c. Then $n = 10(2c) + 14(-c) \in D$ as required.

(l) Induct on n. Base case $n = 1$ has $b_1 = 2^1 + 1$. Inductive step: if $b_n = 2^n + n$ then $b_{n+1} = 2(2^n + n) - n + 1 = 2^{n+1} + (n+1)$ as claimed.

(m) Induct on n. Base cases $n = 1$ and $n = 2$ have $c_1 = 2^{F_1}$ and $c_2 = 2^{F_2}$. Inductive step: if $c_n = 2^{F_n}$ and $c_{n-1} = 2^{F_{n-1}}$ then $c_{n+1} = c_n c_{n-1} = 2^{F_n} 2^{F_{n-1}} = 2^{F_n + F_{n-1}} = 2^{F_{n+1}}$ as required.

(n) Induct on n . Base cases $n = 1$ and $n = 2$ have $d_1 = 2^1$ and $d_2 = 2^2$. Inductive step: if $d_n = 2^n$ and $d_{n-1} = 2^{n-1}$ then $d_{n+1} = 2^n + 2(2^{n-1}) = 2^n + 2^n = 2^{n+1}$ as required.

(o) If $a = 2c + 1$ and $b = 2d + 1$ then $a^2 + b^2 - 2 = 4(c^2 + c + d^2 + d)$, which is divisible by 8 since $c^2 + c = c(c+1)$ is always even as is $d^2 + d$.

(p) Induct on n. Base case $n = 1$ has $25^1 + 7 = 32$ a multiple of 8. Inductive step: if 8 divides $25^n + 7$, then 8 divides $25 \cdot (25^n + 7) - 24 \cdot 7 = 25^{n+1} + 7$.

(q) Note $(2n)(2n+2) = 4n^2 + 4n$ is 1 less than $(2n+1)^2 = 4n^2 + 4n + 1$.

(r) Show the contrapositive. If a, b are not relatively prime so that $d|a$ and $d|b$ for some $d > 1$, then $d^2|a^2$ and $d^2|b^2$ so a^2, b^2 are not relatively prime. Conversely by (d) if p is prime and $p|a^2$ and $p|b^2$ then $p|a$ and $p|b$ so a, b are not relatively prime.

(s) Note $\frac{1}{2^n + k} \geq \frac{1}{2^{n+1}}$ for each $k = 1, 2, \ldots, 2^n$ so sum exceeds $\frac{2^n}{2^{n+1}} = \frac{1}{2}$.

(t) Induct on n. Base case $n = 1$ has $1 > \frac{1}{2}$. Inductive step: suppose $1 + \frac{1}{2} + \cdots + \frac{1}{2^n} > \frac{n}{2}$. Then $1 + \frac{1}{2} + \cdots + \frac{1}{2^{n+1}} = [1 + \frac{1}{2} + \cdots + \frac{1}{2^n}] + [\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}] > \frac{n}{2} + \frac{1}{2} = \frac{n+1}{2}$ by (s).
