E. Dummit's Math 1365 ~ Intro to Proof, Fall 2022 ~ Homework 7, due Tue Nov 1st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation.

   (a) $R_1 = \{(1,1), (2,1), (2,2)\}$ on the set $\{1,2\}$.

   (b) $R_2 = \{(1,1), (2,1), (2,2)\}$ on the set $\{1,2,3\}$.

   (c) $R_3 = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$ on the set $\{1,2,3\}$.

   (d) $R_4 = \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$ on the set $\{1,2,3\}$.

   (e) R_5, the relation on human beings where $a\ R_4\ b$ means "a has the same last name as b".

   (f) R_6, the relation on human beings where $a\ R_5\ b$ means "a is a parent of b".

   (g) R_7, the empty relation on the empty set. (Be very careful with the quantifiers in the definitions of the terms!)

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2. For each relation R on A, identify whether or not R is (i) reflexive, (ii) antisymmetric, (iii) transitive, (iv) a partial ordering, and (v) a total ordering.

   (a) $A = \{a,b,c\}$, $R_8 = \{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}$.

   (b) $A = \mathbb{R}$, $R_9 = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x^2 \le y^2\}$.

   (c) $A = \mathbb{R}$, $R_{10} = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y < x\}$.

   (d) $A = \{4,6,8,10,12\}$ and $R_{11} = \{(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(10,10),(12,12)\}$ is the divisibility relation on A.

   (e) $A = \mathbb{Z}$, $R_{12} = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : b = a \text{ or } b = a+1\}$.

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3. Find all partitions of the set $\{1,2,3\}$ and write down all ordered pairs in the corresponding equivalence relation for each.

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4. Another property of relations that arises on occasion is as follows: we say a relation R on a set A is <u>irreflexive</u> when $a \not R\, a$ for all $a \in A$. This property is essentially the opposite of being reflexive.

   Example: The order relation $<$ on real numbers is irreflexive, because $a < a$ is false for all real numbers a. (In fact this particular relation is one main motivation for considering irreflexive relations, since it is a property held by strict inequalities.)

   (a) For each relation R_1 through R_{12} in problems 1 and 2, identify whether the relation is irreflexive.

   (b) Give an example of a relation that is not reflexive and also not irreflexive. (Thus, being irreflexive is not the same as being not reflexive.)

   (c) Does there exist a relation on $A = \{1,2,3\}$ that is both reflexive and irreflexive? Does there exist any relation on any set that is both reflexive and irreflexive? Explain why or why not.

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5. Suppose that R is a relation on the set A.

   Proposition: If R is symmetric and transitive, then R is reflexive.
   Proof: Let $a \in A$ be arbitrary. Because R is symmetric, if $a\ R\ b$ then $b\ R\ a$. Therefore, applying transitivity to $a\ R\ b$ and $b\ R\ a$ yields $a\ R\ a$. Because a was arbitrary, we conclude $a\ R\ a$ for every $a \in A$, so R is reflexive.

   (a) The proof given above is erroneous. (If it were correct, we would not bother to include reflexivity in the definition of an equivalence relation!) Explain, briefly, what the error in the proof is. [Hint: See problem 6 of homework 2 for inspiration.]

   (b) Construct a counterexample to the proposition using the set $A = \{1, 2\}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. Suppose $R : A \to B$ and $S : A \to B$ are relations (i.e., subsets of $A \times B$). For each statement below, identify whether it is true or false. If it is true then prove it, and if it is false then give a counterexample.

   (a) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

   (b) $(R \cup S)^{-1} = R^{-1} \cap S^{-1}$.

   (c) $R = R^{-1}$ if and only if R is symmetric.

   (d) The inverse relation of a partial ordering R on a set A is also a partial ordering on A.

   (e) The only relation on a set A that is both symmetric and antisymmetric is the identity relation.

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7. The goal of this problem is to show that taking intersections of relations preserves most of their standard properties we have defined. Suppose I is an indexing set and R_i is a relation on the set A for each $i \in I$.

   (a) If each R_i is reflexive, show that $\bigcap_{i \in I} R_i$ is also reflexive.

   (b) If each R_i is symmetric, show that $\bigcap_{i \in I} R_i$ is also symmetric.

   (c) If each R_i is antisymmetric, show that $\bigcap_{i \in I} R_i$ is also antisymmetric.

   (d) If each R_i is transitive, show that $\bigcap_{i \in I} R_i$ is also transitive.

   (e) Deduce that the intersection of an arbitrary collection of equivalence relations is an equivalence relation, and that the intersection of an arbitrary collection of partial orderings is a partial ordering.

   (f) If R is any relation on A, show that R has a well-defined "equivalence closure": namely, a relation $\tilde{R}$ on A such that $R \subseteq \tilde{R}$ where $\tilde{R}$ is an equivalence relation such that $\tilde{R}$ is smallest among all equivalence relations containing R. [Hint: Take the intersection of all equivalence relations containing R. Make sure to show that this intersection is not empty.]

   (g) Illustrate (f) by finding the equivalence closures of the relations $R_1 = \{(1, 2), (1, 3), (2, 4)\}$ and $R_2 = \{(1, 2), (3, 3)\}$ on $A = \{1, 2, 3, 4\}$. [Hint: Identify which elements must go together in each equivalence class.]
