E. Dummit's Math 1365 ~ Intro to Proof, Fall 2022 ~ Homework 5, due Tue Oct 11th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Find the following:

   (a) Find the gcd and lcm of 288 and 600.

   (b) Find the gcd and lcm of $2^8 3^{11} 5^7 7^8 11^2$ and $2^4 3^8 5^7 7^7 11^{11}$.

   (c) Find an ordered pair of integers $(x, y)$ such that $6x + 19y = 1$.

   (d) Find an ordered pair of integers $(x, y)$ such that $89x + 17y = 1$.

   (e) Find the prime factorizations of 999 and 1001.

   (f) Find the prime factorizations of 2023 and $2023^{2022}$.

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear arguments.

2. The Fibonacci-Virahanka numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci-Virahanka sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ....

   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer $n$. [Hint: Use induction.]

   (b) Prove that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer $n$.

   (c) Prove that $F_{n+1}^2 - F_n F_{n+2} = (-1)^n$ for every positive integer $n$.

   (d) A "lyrical pattern" consists of a sequence of long and short beats, where a long beat is twice as long as a short beat. Some examples are long-long-short-long (length 7) and short-short-short-short-long (length 6). Prove that for all $n \geq 1$, the number of lyrical patterns whose length equals $n$ short beats is the Fibonacci number $F_{n+1}$. [Hint: What happens if you delete the last beat in a sequence of length $n$?]

   - Remark: The study of lyrical patterns by Indian poets writing in Sanskrit (e.g., Pingala in approximately 200 BCE) is the first known analysis of the Fibonacci-Virahanka numbers (historically called the Fibonacci numbers following Fibonacci's description of them in 1202 CE, but Virahanka was the first to give a clear description of them in approximately the year 700 CE). There are very many identities involving the Fibonacci numbers, and they show up in many applications.

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3. Prove that $\log_3 5$ is irrational. [Hint: Suppose otherwise, so that $\log_3 5 = a/b$. Convert this to statement about positive integers and find a contradiction.]

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4. The goal of this problem is to study which numbers of the form $N = a^k - 1$ can be prime, where $a$ and $k$ are positive integers greater than 1.

   (a) Show that $x^k - 1$ is divisible by $x - 1$, for any integer $x$.

   (b) Show that if $a > 2$, then $N = a^k - 1$ is not prime.

   (c) Show that if $k$ is composite, then $N = 2^k - 1$ is not prime. [Hint: If $k = rs$, show $2^r - 1$ divides $N$.]

   (d) Conclude that if $N = a^k - 1$ is prime, then $a = 2$ and $k$ is prime. (Such primes are called Mersenne primes.)

   (e) If $p$ is prime, is $2^p - 1$ always prime?

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5. The goal of this problem is to demonstrate that the uniqueness of prime factorizations is not as obvious as it may seem. Let $S$ be a nonempty set of positive integers, and define an <u>$S$-prime</u> to be an element $p \in S$ such that there do not exist $a, b \in S$ such that $ab = p$ and $1 < a, b < p$. (If $S$ is the set of all positive integers, then this definition reduces to the usual one for prime numbers.) Let $E = \{2, 4, 6, 8, 10, \dots\}$ be the set of even positive integers and $O = \{1, 3, 5, 7, 9, 11, \dots\}$ be the set of odd positive integers.

   (a) Which of 2, 4, 6, 8, 10, 12, 14, and 16 are $E$-primes?

   (b) Show that $2n \in E$ is an $E$-prime if and only if $n$ is odd. [Hint: Show the contrapositive.]

   (c) Show that 60 has two different factorizations as a product of $E$-primes. Deduce that $E$ does not have unique $E$-prime factorization.

   (d) Which of 1, 3, 5, 7, 9, 11, 13, 15 are $O$-primes?

   (e) Show that $p \in O$ is an $O$-prime if and only if $p$ is an odd prime integer.

   (f) Explain why $O$ has unique $O$-prime factorization.

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6. The goal of this problem is to prove the rational root test from algebra, and derive some of its consequences.

   (a) Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with integer coefficients, meaning that $a_n, a_{n-1}, \dots, a_0$ are integers. Prove the <u>rational root test</u>: if $r/s$ is a rational root in lowest terms, meaning that $r, s$ are relatively prime and $p(r/s) = 0$, then $r | a_0$ and $s | a_n$. [Hint: Clear denominators and rearrange to show that $s | a_n r^n$ and $r | a_0 s^n$.]

   (b) Suppose $x$ is such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ for some integers $a_{n-1}, \dots, a_0$. Show that if $x$ is not an integer, then $x$ is irrational.

   (c) If $n$ is an integer, prove that $\sqrt{n}$ is irrational unless $n$ is a perfect square.

   (d) Prove that $\sqrt{2}+\sqrt{3}$ is irrational. [Hint: Show $\sqrt{2}+\sqrt{3}$ is not an integer and then consider $(\sqrt{2}+\sqrt{3})^2-5$.]

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