E. Dummit's Math 1365 ∼ Intro to Proof, Fall 2022 ∼ Homework 4, due Tue Oct 4th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Each of the following propositions has an incorrect step in its proof. Identify the error in each case:

   (a) <u>Proposition</u>: For all real numbers x, $x^2 \geq 0$.
   <u>Proof</u>: We prove this result by contradiction. So, assume that the result is false, meaning that for all real numbers x, $x^2 < 0$. But this is false because if $x = 2$, then x^2 is not less than 0. We have reached a contradiction, so we are done.

   (b) <u>Proposition</u>: If $0 \notin A$ and $A \subseteq B$, then $0 \notin B$.
   <u>Proof</u>: Let x be an arbitrary element of A. Since $0 \notin A$, we know that $x \neq 0$. Also, because $A \subseteq B$, we have $x \in B$. Since x was arbitrary and $x \in B$ with $x \neq 0$, we have $0 \notin B$ as claimed.

   (c) <u>Proposition</u>: $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R}, \ y = x(y^2 + 2)$.
   <u>Proof</u>: Let $x = y/(y^2 + 2)$. Then $x(y^2 + 2) = \dfrac{y}{y^2 + 2}(y^2 + 2) = y$ and so $y = x(y^2 + 2)$ as required.

   (d) <u>Proposition</u>: If m is odd and n is even, then $m^2 - n^2 = m + n$.
   <u>Proof</u>: Suppose m is odd and n is even, so that $m = 2k + 1$ for some integer k and $n = 2k$ for some integer k. Then $m^2 - n^2 = (2k+1)^2 - (2k)^2 = 4k^2 + 4k + 1 - 4k^2 = 4k + 1 = m + n$, as claimed.

   (e) <u>Proposition</u>: For any real $x < 0$, there exists a unique $y \in \mathbb{R}$ such that $y^2 = x$ and $y > 0$.
   <u>Proof</u>: Suppose that there exist two values y_1 and y_2 such that $y_1^2 = x$ and $y_2^2 = x$ with $y_1 > 0$ and $y_2 > 0$. Subtracting yields $y_1^2 - y_2^2 = 0$ so that $(y_1 - y_2)(y_1 + y_2) = 0$ so that $y_1 = y_2$ or $y_1 = -y_2$. But since $y_1 > 0$ and $y_2 > 0$ we cannot have $y_1 = -y_2$. Therefore $y_1 = y_2$ which means there exists a unique $y \in \mathbb{R}$ such that $y^2 = x$ and $y > 0$ as claimed.

   (f) <u>Proposition</u>: If $a_1 = 1$ and $a_{n+1} = 2a_n - 1$ for all $n \geq 1$, then $a_n = 2^n + 1$ for all n.
   <u>Proof</u>: We prove this by induction on n. The base case $n = 1$ is trivial. For the inductive step, suppose $a_n = 2^n + 1$. Then $a_{n+1} = 2a_n - 1 = 2(2^n + 1) - 1 = 2^{n+1} - 1$ as required.

   (g) <u>Proposition</u>: If $a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for all $n \geq 1$, then $a_n = 2^n$ for all n.
   <u>Proof</u>: We prove this by strong induction on n. The base case $n = 1$ follows since $a_1 = 2 = 2^1$. For the inductive step, suppose $a_k = 2^k$ for all $k \leq n$. Then $a_{k+1} = 4a_n - 4a_{n-1} = 4 \cdot 2^n - 4 \cdot 2^{n-1} = 4 \cdot 2^n - 2 \cdot 2^n = 2 \cdot 2^n = 2^{n+1}$ as required.

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2. For each pair of integers (a, b), use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ AND also to find integers x and y such that $d = ax + by$. (Make sure to include the Euclidean algorithm calculations in your writeup.)

   (a) $a = 12$, $b = 44$.
   (b) $a = 20$, $b = 107$.
   (c) $a = 2022$, $b = 20232$.
   (d) $a = 5567$, $b = 12445$.
   (e) $a = 233$, $b = 144$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Prove the following properties of divisibility:

    (a) If a, b are integers then $a|b$ if and only if $(-a)|b$.

    (b) If a, b, c, x, y are integers with $a|b$ and $a|c$, show that $a|(xb + yc)$.

    (c) If a, b, m are integers with $m \neq 0$, show that $a|b$ if and only if $(ma)|(mb)$.

    (d) If a, b are integers with $a|b$ and $b|a$ then $a = b$ or $a = -b$. [Hint: Use the fact that the only divisors of 1 are 1 and -1. Be careful when $a = 0$ or $b = 0$.]

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4. Show the following:

    (a) For all positive integers n, show that the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$ equals $\dfrac{n(n+1)(2n+1)}{6}$.

    (b) For all positive integers n, show that the sum $3^0 + 3^1 + 3^2 + \cdots + 3^n$ equals $\dfrac{3^{n+1} - 1}{2}$.

    (c) The sequence $\{d_n\}_{n \geq 1}$ is defined recursively by $d_1 = 3$, $d_2 = 9$, and for all $n \geq 3$, $d_n = 2d_{n-1} + 3d_{n-2}$. Prove that $d_n = 3^n$ for all positive integers n.

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5. Recall that an integer n is <u>even</u> if $n = 2a$ for some integer a, while an integer n is <u>odd</u> if $n = 2b + 1$ for some integer b. As noted in class, as a consequence of the division algorithm, every integer is either even or odd, and no integer is both.

    (a) Show that the sum of two even or two odd integers is even, and that the sum of an even integer and an odd integer is odd.

    (b) Show that the product of an even integer with any integer is even, and the product of two odd integers is odd.

    (c) Show that if n is even then n^2 is even, and if n is odd then n^2 is odd.

    (d) Deduce that n^2 is even if and only if n is even. [Hint: What is the contrapositive of "if n is odd then n^2 is odd"?]

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6. The goal of this problem is to prove the Binomial Theorem. First, we define the factorial function as $n! = n \cdot (n-1) \cdots 2 \cdot 1$, so that for example $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and so forth. Now define the <u>binomial coefficient</u> $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$, and note that $\binom{n}{0} = \binom{n}{n} = 1$ for every n.

    (a) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for every $0 \leq k \leq n$. Conclude in particular that $\binom{n}{k}$ is always an integer.

    (b) Suppose that x and y are arbitrary real numbers. Prove the Binomial Theorem: that $(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n = \sum_{k=0}^{n} \binom{n}{k}x^{n-k}y^k$ for any positive integer n. [Hint: Induct on n.]

    - <u>Remark</u>: The binomial coefficient $\binom{n}{k}$ counts the number of ways of selecting a subset of k elements from the set $\{1, 2, 3, \ldots, n\}$. Intuitively, for (b), in distributing out the product $(x + y)^n = (x + y)(x + y) \cdots (x + y)$, a term $x^{n-k}y^k$ is formed when we select a y from exactly k of the terms. So the total number of ways to obtain a term $x^{n-k}y^k$ is the same as the number of ways of selecting a subset of k elements from $\{1, 2, 3, \ldots, n\}$, and there are $\binom{n}{k}$ such subsets.