E. Dummit's Math 1365 $\sim$ Intro to Proof, Fall 2022 $\sim$ Homework 3, due Tue Sep 27th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $U = \mathbb{R}_{\geq 0}$, the nonnegative real numbers, and take $P(x, y)$ to be the statement "$x < y$".
   (a) There are 8 possible ways, listed below, of quantifying both x and y in the statement $P(x, y)$. For each statement, translate it into words and then find its truth value.
      i. $\forall x \forall y,\ P(x, y)$
      ii. $\forall x \exists y,\ P(x, y)$
      iii. $\exists x \forall y,\ P(x, y)$
      iv. $\exists x \exists y,\ P(x, y)$
      v. $\forall y \forall x,\ P(x, y)$
      vi. $\forall y \exists x,\ P(x, y)$
      vii. $\exists y \forall x,\ P(x, y)$
      viii. $\exists y \exists x,\ P(x, y)$
   (b) Repeat part (a) with $P(x, y)$ replaced by the statement "$y = 2x$" and with universe $\mathbb{Z}$.
   (c) Of the eight quantified statements in part (a), two pairs will always be logically equivalent for any statement $P(x, y)$. Identify these pairs.

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2. For each statement, translate it into words and then find its truth value. (Assume that all capital-letter variables refer to sets.)
   (a) $\forall x \in \mathbb{R},\ x^2 > 0$.
   (b) $\exists x \in \mathbb{Z},\ x^2 - 3x + 2 = 0$.
   (c) $\forall A \forall B \forall C,\ [x \in A \cap B \cap C] \Rightarrow [x \in A \cap B] \wedge [x \in A \cup C]$.
   (d) $\forall A \exists x (x \in A)$.
   (e) $\forall A \exists x \exists y,\ (A = \emptyset) \vee [(x \in A) \wedge (y \in A)]$.

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3. Negate each given statement and then rewrite the result as an equivalent positive statement. (All quantifiers should appear ahead of any negation operators.)
   (a) $\exists x,\ x^2 = 2$.
   (b) $\exists x \exists y,\ x + y \neq 5$.
   (c) $\forall x \exists y \exists z,\ x \cdot y + z > 2$.
   (d) $\forall a \in A\, \exists b \in B\, (a \in C \wedge b \in C)$.

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4. Assume that the following two statements are both true: 1) Pinocchio always lies, and 2) Pinocchio says "All my hats are green". For each statement, determine whether it must be true, must be false, or could be either true or false:
   (a) Pinocchio has at least one green hat.
   (b) Pinocchio has no green hats.
   (c) Pinocchio has no hats.
   (d) Pinocchio has at least one hat.
   (e) Pinocchio has exactly one green hat.

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5. Let $I = \{1, 2, 3, \dots\}$ be the set of positive integers, and for each $i \geq 1$ let $F_i = \{i, 2i, 3i, 4i, \dots\}$ be the set of positive integer multiples of i. Find $\bigcup_{i \in I} F_i$ and $\bigcap_{i \in I} F_i$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. Many of our proofs involving sets have implicitly used quantifiers. The goal of this problem is to make some of these ideas more explicit by analyzing a portion of the proof (given in class) that if A and B are any sets then $A \cap B = A$ implies that $A \subseteq B$.

   (a) If P and Q are propositions, show that $P \Rightarrow (P \wedge Q)$ is logically equivalent to $P \Rightarrow Q$.

   (b) If A and B are sets, show that $\forall x, (x \in A) \Rightarrow [(x \in A) \wedge (x \in B)]$ is logically equivalent to $\forall x, (x \in A) \Rightarrow (x \in B)$. [Hint: Use (a).]

   (c) Explain why (b) says that $A \subseteq A \cap B$ is equivalent to $A \subseteq B$.

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7. The goal of this problem is to examine the quantifier "there exists a unique", written as $\exists!$. Thus, for example, the statement "there exists a unique x such that $x^2 = 2$" would be written $\exists! x,\ x^2 = 2$. The meaning of this quantifier is that there exists an element x satisfying the hypotheses, and that there is exactly one such x.

   (a) Identify the truth values of the following statements:

      i. $\exists! n \in \mathbb{Z},\ n^2 = 2$.
      ii. $\exists! n \in \mathbb{Z},\ n^2 = 4$.
      iii. $\exists! n \in \mathbb{Z}_+,\ n^2 = 4$.
      iv. $\exists! n \in \mathbb{Z},\ n^2 = 0$.
      v. $\forall x > 0\, \exists! y > 0,\ xy = 1$, with universe $\mathbb{R}$.

   (b) It may seem that $\exists!$ is a new quantifier, but in fact, it can be expressed in terms of $\exists$ and $\forall$. Explain why $\exists! x \in A,\ P(x)$ is logically equivalent to $\exists x \in A,\ P(x) \wedge [\forall y \in A,\ P(y) \Rightarrow (y = x)]$ for any proposition $P(x)$. (Your explanation does not have to be fully rigorous.)

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8. Suppose that $\mathcal{A} = \{A_i : i \in I\}$ and $\mathcal{B} = \{B_j : j \in J\}$ are two families of sets indexed by I and J respectively.

   (a) Prove that for each $s \in I$ it is true that $\bigcap_{i \in I} A_i \subseteq A_s$.

   (b) Prove that for each $t \in J$ it is true that $B_t \subseteq \bigcup_{j \in J} B_j$.

   (c) Prove that if $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ then $\bigcap_{i \in I} A_i \subseteq \bigcup_{j \in J} B_j$. [Hint: Suppose $S \in \mathcal{A} \cap \mathcal{B}$ and apply (a) and (b) to it.]
