E. Dummit's Math 1365 ~ Intro to Proof, Fall 2022 ~ Homework 2, due Tue Sep 20th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

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1. Suppose $A = \{1, 3, 9\}$, $B = \{6, 7, 8, 9\}$, and $C = \{2, 3, 5, 7\}$, with universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Calculate the following (no justification is required):
   (a) $A \cap B$
   (b) $A \cup C$
   (c) $(A \cap B) \cap C$
   (d) $A \cap (B \cap C)$
   (e) $A \cap (B \cup C)$
   (f) $(A \cap B) \cup C$
   (g) $B^c \cap C^c$
   (h) $(A^c \cup B) \cap (B^c \cup C)$
   (i) $A \times B$
   (j) $B \times A$
   (k) $(A \times B) \cap (B \times A)$
   (l) $(A \cap B) \times (B \cup C)$

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2. Suppose $A = \{1, 2, \{1\}, \emptyset\}$. Identify each of the statements below as true or as false, and give a brief (1-sentence) explanation why.
   (a) $\emptyset \in A$.
   (b) $\emptyset \subseteq A$.
   (c) $\{1\} \in A$.
   (d) $\{1\} \subseteq A$.
   (e) $\{1, 2\} \in A$.
   (f) $\{1, 2\} \subseteq A$.
   (g) $\{\{1\}\} \in A$.
   (h) $\{\{1\}\} \subseteq A$.

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3. In some mathematical arguments, we frequently need to establish that two sets S and T are equal. A very natural way to do this is to show that $S \subseteq T$ (by establishing that $x \in S$ implies $x \in T$) and also that $T \subseteq S$ (by establishing that $x \in T$ implies $x \in S$). Using this method, or otherwise, and supposing that A, B, and C are any sets contained in a universal set U, provide a rigorous proof of each of the following statements (in particular, you may *not* appeal to Venn diagrams in your solutions and must use *only* formal properties of set containments):
   (a) Prove that the set $S = \{0^2, 2^2, 4^2, 6^2, \dots\}$ of squares of even integers is equal to the set $T = \{4 \cdot 0^2, 4 \cdot 1^2, 4 \cdot 2^2, \dots\}$ of numbers of the form $4k^2$ where k is an integer.
   (b) Prove that $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.
   (c) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

   - Remark: Note that some of these parts ask for a proof of a result stated but not explicitly proven in the course notes. As such, you may not quote those results in the responses (e.g., your response to part (c) cannot be "This follows immediately from the distributive laws for Cartesian products"), because it would be circular logic.

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4. Suppose A and B are sets. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

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5. In some mathematical arguments, there can arise several different possible cases. In such situations, it can be useful to break the argument apart and analyze each of the possible cases separately, rather than trying to deal with them all at once. Indeed, the idea of breaking into cases is the procedure underlying the use of truth tables: we simply evaluate logical expressions in all possible cases to see whether they agree. Using this idea or otherwise, prove the following:
   (a) If A, B, C are any sets, prove that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
   (b) If A, B, C are any sets, prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, and deduce in fact that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (c) If A, B are any subsets of a universal set U, prove that $(A \cup B)^c = A^c \cap B^c$. [Hint: For any $x \in U$, there are four possible cases: either $x \in A, x \in B$ or $x \in A, x \notin B$ or $x \notin A, x \in B$ or $x \notin A, x \notin B$.]

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6. Suppose A and B are sets. The goal of this problem is to study the question of when $A \times B = B \times A$.

   Proposition: $A \times B = B \times A$ if and only if $A = B$.
   Proof: If $A = B$, then clearly $A \times B = A \times A = B \times A$. Now suppose $A \times B = B \times A$, and let $a \in A$ and $b \in B$ be arbitrary elements of A and B respectively. Then by definition, $(a, b) \in A \times B$, and so by hypothesis, $(a, b) \in B \times A$. This means $a \in B$ and $b \in A$. Since $a \in A$ and $b \in B$ are arbitrary, the fact that $a \in B$ implies $A \subseteq B$, and the fact that $b \in A$ implies $B \subseteq A$. We conclude that $A = B$, as required.

   (a) Consider the proposition and proof given above. Show that the proposition is incorrect by explaining why taking $A = \emptyset$ and $B = \{1, 2\}$ yields a counterexample.

   (b) Part (a) shows that the proposition stated above is incorrect, so the proof must contain a logical error. Identify what the error is, and why it causes the proof to be incorrect. [Hint: The counterexample from part (a) is clearly relevant.]

   (c) Give a corrected version of the proposition, and then give a correct proof. [Hint: Your corrected proposition should start with "$A \times B = B \times A$ if and only if $A = B$ or..." and the proof should also make sure to address the error you identified in part (b).]

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7. In addition to union and intersection, there are a few other set operations that arise from time to time. Two of these are the set difference $A \backslash B = \{x \in A : x \notin B\}$, the set of elements of A not in B, and the symmetric difference $A \Delta B = (A \backslash B) \cup (B \backslash A)$, the elements in either A or B but not both. (Observe that $A \Delta B = B \Delta A$, whence the name *symmetric* difference.)

   (a) If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$, and $C = \{1, 3, 5\}$, find $A \backslash B$, $B \backslash A$, $A \backslash C$, $C \backslash A$, $B \backslash C$, and $C \backslash B$.

   (b) If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$, and $C = \{1, 3, 5\}$, find $A \Delta B$, $A \Delta C$, and $B \Delta C$.

   (c) If A and B are subsets of a universal set U, show that $A \backslash B = A \cap B^c$. Deduce that $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$.

   (d) For each of the following statements, decide whether it is true or whether it is false. Then prove the true statements (you may use an informal Venn diagram argument) and give an explicit counterexample to the false statements:

      i. $(A \backslash B) \cup B = A$
      ii. $(A \backslash B) \cup (A \backslash B^c) = A$
      iii. $A \Delta B = (A \cup B) \backslash (A \cap B)$
      iv. $(A \backslash B) \cap (B \backslash A) = \emptyset$
      v. $(A \Delta B)^c = A^c \backslash B$
      vi. $(A \backslash B) \backslash C = (A \backslash C) \backslash B$
      vii. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
      viii. $(A \cap B) \Delta C = (A \Delta C) \Delta (A \backslash B)$

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8. Wikipedia has many articles. Some of these articles are lists, such as the article "List of American mathematicians". There are even articles which list other lists, such as the article "List of lists of mathematical topics". Some of these lists of lists contain themselves, such as the article "List of lists of lists".

   (a) Consider the Wikipedia article titled "List of lists that do not contain themselves". Can this article be listed on itself? (Assume, as most students do, that everything on Wikipedia is accurate.)

   (b) The only true test of a hypothesis is empirical evidence. To this end, what *actually happens* when you visit the Wikipedia article "List of lists that do not contain themselves"?
