E. Dummit's Math 1365 ∼ Intro to Proof, Fall 2022 ∼ Homework 9, due Thu Dec 1st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. In class, we showed using pigeonhole ideas that if A is a finite set and $f : A \to A$ is a function, then f is one-to-one if and only if f is onto. The goal of this problem is to show via example that both implications are FALSE in the situation where A is an infinite set.

   (a) Find an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that is one-to-one but not onto.

   (b) Find an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that is onto but not one-to-one.

   (c) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.

   (d) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one.

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2. Determine whether each of the following sets is finite, countably infinite, or uncountably infinite:

   (a) The set of irrational real numbers.

   (b) The set of real numbers that can be described using at most 60 characters written in English. (For example, 1 can be described using the string "one" while π can be described as "the ratio of a circle's circumference to its diameter".)

   (c) The set of real numbers that can be described using at most 100,000 characters written in English.

   (d) The set of real numbers that can be described using a finite number of characters written in English.

   (e) The set of real numbers that cannot be described by any finite string of characters written in English.

   - Remark: The ideas from (b)-(e) can also be used to discuss the cardinality of the "computable numbers" (real numbers that can be computed to arbitrarily good accuracy by a finite, terminating algorithm).

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. The goal of this problem is to give another proof that $\mathbb{Q}$ is countable.

   (a) Let $f : \mathbb{Q} \to \mathbb{Z}_{>0}$ be the map defined by $f((-1)^k a/b) = 2^k 3^a 5^b$, where $a \geq 0$, $b > 0$, a/b is in lowest terms, and $k \in \{0, 1\}$. Show that f is one-to-one.

   (b) Show that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Z}_{>0}$. [Hint: Use Cantor-Schröder-Bernstein with $g : \mathbb{Z}_{>0} \to \mathbb{Q}$ defined by $g(n) = n$.]

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4. The goal of this problem is to show that there exists a bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$.

   (a) Show that there exists a one-to-one map $f : [0, 1] \to [0, 1] \times [0, 1]$.

   (b) Consider the map $g : [0, 1] \times [0, 1] \to [0, 1]$ where $g(0.d_1 d_2 d_3 \dots, 0.e_1 e_2 e_3 \dots) = 0.1d_1 e_1 1d_2 e_2 1d_3 e_3 \dots$, where we always choose the decimal expansion ending in a string of 9s if there is a choice. Show that g is one-to-one.

   (c) Deduce that there exists a bijection between $[0, 1]$ and $[0, 1] \times [0, 1]$.

   (d) Show that there exists a bijection between $[0, 1]$ and $\mathbb{R}$. [Hint: Use $f(x) = x$ and $g(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$.]

   (e) Deduce that there exists a bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$. [Hint: Apply (d) to each copy of $[0, 1]$ in (c).]

   - Remark: Some other (surprising!) results related to these are (i) there exists a *continuous* onto function $f : [0, 1] \to [0, 1] \times [0, 1]$, but (ii) there does not exist a continuous bijection $f : [0, 1] \to [0, 1] \times [0, 1]$. Functions with the property (i) are often called space-filling curves. Also, (iii) there exists an additive bijection $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathbb{R}$.

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5. Using the pigeonhole principle or otherwise, solve the following:

   (a) There are 37 students in Math 1365. Show that at least 4 students were born during the same month of the year.

   (b) Show that if we select 100 different integers, then some pair of them must have difference divisible by 97.

   (c) Show that if 51 elements from the set $\{1, 2, \ldots, 100\}$ are selected, then at least one pair of consecutive integers must be among the chosen 51 elements.

   (d) Show that if each square in a 3×9 grid is colored either red or blue, then there must exist four squares forming a rectangle whose vertices are all the same color. [Hint: How many ways are there to color the 3 squares in a single column?]

   (e) Show that if 10 points are chosen inside a square of side length 3, at least one pair of points must be within a distance $\sqrt{2}$ of one another. [Hint: Cut up the square.]

   (f) Show that if 100 distinct integers are chosen, there exists a subset S of 10 of these integers such that the difference between any pair of integers in S is divisible by 11.

   (g) Let A be a set of n integers. Prove that A contains a nonempty subset whose sum of elements is divisible by n. [Hint: Consider the sums a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, ... , $a_1 + \cdots + a_n$ modulo n. If two of them are the same, subtract them.]

   (h) [Optional Bonus] Suppose five points are drawn on the surface of a sphere. Prove that at least four of the points must lie on some closed hemisphere. [Hint: Choose two points and cut the sphere along the great circle that contains them. Where can the other three points be?]

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6. Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, and let $m > 1$ be a modulus. The goal of this problem is to prove that the Fibonacci numbers are periodic modulo m, meaning that there exists some a such that $F_{n+a} \equiv F_n \pmod{m}$ for all $n \geq 1$.

   (a) Verify the result by finding the period of the Fibonacci numbers mod 2, mod 3, and mod 5. [Hint: Once the values mod m restart at 0, 1, 1 you can identify the period: why?]

   (b) Show that there must exist positive integers $b < c$ such that $F_b \equiv F_c \pmod{m}$ and $F_{b+1} \equiv F_{c+1} \pmod{m}$. [Hint: There are only finitely many possible pairs (F_b, F_{b+1}) modulo m.]

   (c) Suppose $F_x \equiv F_y \pmod{m}$ and $F_{x+1} \equiv F_{y+1} \pmod{m}$. Show that $F_{x+2} \equiv F_{y+2} \pmod{m}$ and also that $F_{x-1} \equiv F_{y-1} \pmod{m}$.

   (d) Suppose $F_b \equiv F_c \pmod{m}$ and $F_{b+1} \equiv F_{c+1} \pmod{m}$ and let $a = c - b$. Prove that $F_{n+a} \equiv F_n \pmod{m}$ for all positive integers n. [Hint: Let S be the set of n for which $F_{n+a} \equiv F_n \pmod{m}$. Using part (c), and the fact that $b \in S$ and $b + 1 \in S$, explain why S must be the set of all positive integers.]

   (e) Deduce that the Fibonacci numbers are periodic modulo m.

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7. [Optional] Suppose B is a subset of A. The goal of this problem is to formalize an alternative solution to problem 3a from midterm 2, whose idea is as follows: "If R is an equivalence relation on A and B is a subset of A, then R corresponds to a partition of A. Intersecting this partition with B yields a partition of B, which then corresponds to the equivalence relation $R \cap (B \times B)$ on B."

   (a) Suppose $\mathcal{P}_A = \{A_i\}_{i \in I}$ is a partition of A and define $B_i = A_i \cap B$. Show that $\mathcal{P}_B = \{B_i\}_{i \in I}$ is a partition of B.

   (b) Suppose R is an equivalence relation on A. Show that $S = R \cap (B \times B)$ is an equivalence relation on B. [Hint: Use (a).]

   - <u>Remark</u>: There were very few correct solutions to problem 3a on the midterm. You may find it instructive to compare this approach to the direct method of showing $R \cap (B \times B)$ is reflexive, symmetric, and transitive.
