E. Dummit's Math 1365 ~ Intro to Proof, Fall 2022 ~ Homework 1, due Tue Sep 13th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

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1. Each item below contains a proposition and a claimed proof of the proposition. Decide whether each proof is correct, and for the incorrect proofs, identify (briefly!) at least one mistake or error.

   (a) Proposition: The integer 2 is odd.
   Proof: It is a fact about odd integers that any odd integer plus any odd integer always gives an even integer. Because $2+2=4$, and 4 is an even integer, this means 2 must be odd.

   (b) Proposition: If x is an integer and $3x-2=7$, then $x=3$.
   Proof: Suppose $x=3$. Then $3x-2=3(3)-2=7$. Therefore, if $3x-2=7$, then $x=3$.

   (c) Proposition: If a and b are integers with $a=b$, then $a=0$.
   Proof: Suppose $a=b$. Multiplying by a yields $a^2=ab$ and then subtracting b^2 yields $a^2-b^2=ab-b^2$. Factoring yields $(a-b)(a+b)=(a-b)b$ and cancelling yields $a+b=b$. Finally, subtracting b yields $a=0$ as claimed.

   (d) Proposition: Every odd number greater than 1, except for 9, is prime.
   Proof: Clearly, 3 is prime, 5 is prime, 7 is prime, 9 is not prime, 11 is prime, and 13 is prime. Since we have excluded 9, all odd numbers greater than 1 are prime.

   (e) Proposition: If x is an integer with $x \neq 1$ and $\dfrac{x^2-1}{x-1}=3$, then $x=2$.
   Proof: If $x \neq 1$, then $\dfrac{x^2-1}{x-1}=\dfrac{(x-1)(x+1)}{x-1}=x+1$, so $\dfrac{x^2-1}{x-1}=3$ implies $x+1=3$. By subtracting 1 from both sides of the equality, we conclude that $x=2$.

   (f) Proposition: For an integer n, if n^2 is odd then n is odd.
   Proof: We equivalently show that if n is not odd then n^2 is not odd. If n is not odd, which is to say, n is even, then $n=2k$ for some integer k. Then $n^2=4k^2$ is even, which in particular means n^2 is not odd, as desired.

   (g) Proposition: For an integer m, m is even if and only if m^2 is even.
   Proof: Suppose m is even. Then $m=2k$ for some integer k, meaning that $m^2=(2k)^2=4k^2=2\cdot 2k^2$ and thus m^2 is even. Therefore, m is even if and only if m^2 is even.

   (h) Proposition: Suppose that x and y are integers and $x \neq 2$. If $x^2y=4y$, then $y=0$.
   Proof: Suppose $x^2y=4y$. Then $(x^2-4)y=0$. Since $x \neq 2$, that means $x^2 \neq 4$, so $x^2-4 \neq 0$. Then we can cancel the nonzero term x^2-4 from $(x^2-4)y=0$ to conclude that $y=0$. Therefore, if $x^2y=4y$, then $y=0$.

   (i) Proposition: The set of integers divisible by both 4 and 6 is the same as the set of multiples of 12.
   Proof: Every multiple of 12 is automatically a multiple of 4 and a multiple of 6. Therefore, the sets of multiples of 4 and 6 are multiples of 12.

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2. Find a counterexample to the following statement: if p is a prime number, then 2^p-1 is also a prime number.

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3. After discussing Euclid's proof that there are infinitely many primes, it is frequently claimed (occasionally in actual textbooks) that if p_1, p_2, ... , p_k is a list of the first k primes, then the number $p_1p_2\cdots p_k+1$ is always prime. Show that this statement is false by giving an explicit counterexample. (You may want to use a computer!)

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4. Write explicitly the converse, inverse, and contrapositive of the following conditional statements:

   (a) If you do not study for your exams, then you will get bad grades.

   (b) If you want to bake a cake, then you must have eggs and flour.

   (c) If n is an odd integer greater than 7, then n is the sum of three odd primes.

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5. Suppose that P, Q, and R are any propositions.

   (a) Prove that if P and $P \Rightarrow Q$ are both true, then Q is also true. [Hint: Use a truth table to identify all cases in which both P and $P \Rightarrow Q$ are true.]

   (b) Suppose that the statements "If it is raining, then it is cloudy" and "It is raining" are both true. Is the statement "It is cloudy" necessarily true? Explain. [Hint: Use (a).]

      - Remark: The result here is one form of reasoning we frequently apply in deductive proofs (formally, it is known as *modus ponens*). The point of this problem is to verify that this form of deduction is, in fact, always logically correct.

   (c) Prove that if $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is also true.

   (d) Suppose that the statements "If it is raining, then it is cloudy" and "If it is cloudy, then people want to stay home" are both true. Is the statement "If it is raining, then people want to stay home" necessarily true? Explain.

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6. Using a truth table or otherwise, determine whether each of following pairs of statements are equivalent. For those that are false, give an explicit counterexample (i.e., truth values for the propositions showing the statements are different):

   (a) $A \wedge (A \vee B)$ and A.

   (b) $\neg(A \vee \neg B) \Rightarrow \neg B$ and $B \Rightarrow A$.

   (c) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ and $P \Rightarrow (P \Leftrightarrow Q)$.

   (d) $\neg(\neg P \wedge Q) \vee (P \vee R) \vee (Q \wedge \neg R)$ and True.
