1. Calculations:
   (a) 5 socks, by pigeonhole.
   (b) 7 socks, by pigeonhole (take out the first pair and then apply to the 5 remaining socks).
   (c) It is not a group, the operation is not associative and the identity is not two-sided.
   (d) $(sr)(sr^2) = s(sr^{-1})r^2 = r$ and $s^2r^3s^4r^5 = r^3r^5 = r^8$.
   (e) $(r^3)^{-1} = r^9$ and $(sr^2)^{-1} = r^{-2}s^{-1} = r^{-2}s = sr^2$.
   (f) This permutation is $(1\,8)(2\,7)(3\,6)(4\,5)$.
   (g) This permutation is $(1\,3\,2\,6\,4\,5)(7) = (1\,3\,2\,6\,4\,5)$.
   (h) $(3\,1\,4)(1\,5) = (1\,5\,4\,3)$ by tracing right to left.
   (i) $(2\,7\,1\,8) \cdot (2\,8) \cdot (1\,8) \cdot (2\,8) = (1\,7)(2\,8)$ by tracing right to left.
   (j) $[(1\,4\,2\,8\,5)(6\,7)]^{-1} = (6\,7)^{-1}(1\,4\,2\,8\,5)^{-1} = (7\,6)(5\,8\,2\,4\,1) = (1\,5\,8\,2\,4)(6\,7)$.
   (k) $\mathbb{Z}/8\mathbb{Z}$ and $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ all work.
   (l) The dihedral group $D_{2\cdot 10}$ of order 20 is non-abelian.
   (m) The groups $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{Q}\backslash\{0\}, \cdot)$ are all countably infinite.
   (n) A Cartesian product of a group in (m) with S_3 or $D_{2\cdot 4}$ is countably infinite and non-abelian.
   (o) The groups $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash\{0\}, \cdot)$ are both uncountably infinite.
   (p) A Cartesian product of one of the groups in (o) with S_3 or $D_{2\cdot 4}$ is uncountably infinite and non-abelian.
   (q) s has order 2, r has order 10, r^2 has order 5, r^3 has order 10.
   (r) $(1\,2\,3)$ has order 3, $(4\,5)$ has order 2, $(1\,2\,3)(4\,5)$ has order 6.
   (s) $(2\,4\,5)$ has order 3, $(1\,5)(2\,3)$ has order 2, $(2\,4\,5) \cdot (1\,5)(2\,3) = (1\,2\,3\,4\,5)$ has order 5.
   (t) The least upper bound is $0.33333\cdots = 1/3$.
   (u) This requires a set with no upper bound, such as $(0, \infty)$ or $\mathbb{R}$ itself.

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2. Proofs:
   (a) Divide the circle into four quarter-circles. By pigeonhole two points must lie in the same quarter-circle of radius 1, and they must be within a distance $\sqrt{2}$ of each other.
   (b) By pigeonhole, since there are n^n functions from $\{1, 2, 3, \dots, n\} \to \{1, 2, 3, \dots, n\}$, some of $\{f^1, f^2, f^3, \dots\}$ must be equal as functions: then $f^A = f^B$ means $f^A(i) = f^B(i)$ for all $i \in \{1, 2, \dots, n\}$.
   (c) By pigeonhole, selecting $n+1$ elements must pick two from one of the n sets $\{1, 2\}$, $\{3, 4\}$, ... , $\{2n-1, 2n\}$.
   (d) Both $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$ are countably infinite, so there is a bijection between these sets since they are both in bijection with the positive integers.
   (e) The functions $f : (0, 1) \to [0, 1]$ with $f(x) = x$ and $g : [0, 1] \to (0, 1)$ with $g(x) = (x + 1)/3$ are both one-to-one, so by Cantor-Schröder-Bernstein there exists a bijection between $(0, 1)$ and $[0, 1]$.
   (f) Induct on n. Base case $n = 1$ is given. For inductive step suppose $gh^n = h^n g$. Then $gh^{n+1} = (gh)(h^n) = (hg)h^n = h(gh^n) = h(h^n g) = h^{n+1}g$ using $gh = hg$ and $gh^n = h^n g$.
   (g) Multiply $g^{-1}h^{-1} = h^{-1}g^{-1}$ on the left by hg and on the right by gh. This yields $hg(g^{-1}h^{-1})gh = hg(h^{-1}g^{-1})gh$. Then $hg(g^{-1}h^{-1})gh = hgg^{-1}h^{-1}gh = hh^{-1}gh = gh$ while $hg(h^{-1}g^{-1})gh = hgh^{-1}g^{-1}gh = hgh^{-1}h = hg$, so $gh = hg$.
   (h) By hypothesis $g^n = e$. Multiplying by g^{-1} on both sides yields $g^{-1}g^n = g^{-1}e = g^{-1}$ and since $g^{-1}g^n = g^{-1}g(g^{n-1}) = eg^{n-1} = g^{n-1}$ we see $g^{n-1} = g^{-1}$.
   (i) The function f is an element of the symmetric group S_n. By Lagrange's theorem, its order divides $n!$ hence is finite. But if the order is A then this means f^A is the identity, which is to say, $f^A(i) = i$ for each $i \in \{1, 2, 3, \dots, n\}$.
   (j) Reflexive: $e \in H$ and $g_1 = eg_1$ so $g_1 \, R \, g_1$. Symmetric: If $g_1 \, R \, g_2$ so that $g_1 = hg_2$ with $h \in H$ then $h^{-1}g_1 = g_2$ and $h^{-1} \in H$, so $g_2 \, R \, g_1$. Transitive: If $g_1 \, R \, g_2$ and $g_2 \, R \, g_3$ so that $g_1 = hg_2$ and $g_2 = kg_3$ with $h, k \in H$ then $g_1 = hg_2 = hkg_3$ and $hk \in H$ so $g_1 \, R \, g_3$.
   (k) First $e \in S$ since $e^2 = e$. Second if $g, h \in S$ then $g^2 = e$ and $h^2 = e$ so $(gh)^2 = ghgh = g^2h^2 = ee = e$ since $gh = hg$ because G is abelian, so $gh \in S$. Finally if $g \in S$ then $g^2 = e$ so $(g^{-1})^2 = (g^2)^{-1} = e$ so $g^{-1} \in S$.
