Syllabus for E. Dummit's Math 7315 ∼ Fall 2021
Number Theory in Function Fields: From Fermat's Last Theorem to the Riemann Hypothesis

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Lecture: Monday-Thursday 6:00pm–7:30pm, Ryder Hall 159.

Instructor: Evan Dummit, Lake Hall 571, edummit@northeastern.edu.

Office Hours: Wednesday 4:00pm-5:30pm + Monday-Thursday 4:00pm-5:00pm, or by appointment.

Course Webpage: https://web.northeastern.edu/dummit/teaching_fa21_7315.html.

Course Textbook: Rosen's "Number Theory in Function Fields". I will be loosely following the textbook in some places and will supplement it with material from elsewhere on some occasions.

Expected Background: There are no formal prerequisites, but students should have comfort with algebra and number theory at the level of Math 4527 or 5111 or 5112. I will freely refer to some results from elementary number theory, commutative algebra, algebraic geometry, Galois theory, and complex analysis, but the goal is to make the course as self-contained as possible.

Course Overview: Classical elementary number theory consists of studying the properties of arithmetic in $\mathbb{Q}$ and its associated ring of integers $\mathbb{Z}$ (e.g., prime factorization, Fermat's little theorem, quadratic reciprocity, the prime number theorem, etc.). Algebraic and analytic number theory then expand this discussion to algebraic extensions K of $\mathbb{Q}$ and their associated rings of integers $\mathcal{O}_K$ (e.g., unique factorization into prime ideals, Dirichlet's unit theorem, zeta functions and L-functions, etc.). The primary aim of this course is to discuss the analogous stories of elementary number theory in the setting of the function field $\mathbb{F}_q(t)$, its "ring of integers" $\mathbb{F}_q[t]$, and of algebraic and analytic number theory in finite extensions of $\mathbb{F}_q(t)$. There are a great many analogies between number theory over algebraic number fields and over algebraic function fields, and our goal is to elucidate (as much as possible) the connections between the two. Much of this material is rarely treated in graduate-level courses, despite its simplicity and its importance to modern number theory and algebraic geometry.

Course Topics: This course is billed as "From Fermat's Last Theorem to the Riemann Hypothesis" and this billing is meant quite literally: on the first day of class we will prove Fermat's Last Theorem, and on the last day of class we will prove the Riemann hypothesis. (Of course, these will be done in the function field setting!)
We begin with a treatment of elementary number theory in $\mathbb{F}_q[t]$, studying primes and factorization, zeta functions, the power reciprocity law, and the analogue of Dirichlet's theorem on primes in arithmetic progression. We will then discuss algebraic function fields, valuations and primes, the dictionary between function fields and algebraic curves, differentials and divisors, Riemann-Roch, zeta functions, Galois theory of function fields, class groups, elliptic curves and elliptic function fields, cyclotomic function fields, L-functions, S-units, and other related topics as interest dictates. We will finish up with a treatment of the Weil conjectures for curves, culminating in a proof of the *abc* conjecture for function fields along with a proof of the Riemann hypothesis.

Grades: Grades will be based on attendance and participation (70%), and on occasional homework assignments (30%), which will be collaborative and may involve roundtable discussions of the solutions. Depending on student and instructor interest, there may also be student presentations on course-related topics during the semester.

Course Schedule: The course is tentatively organized as follows:

Weeks 1-4: Elementary Number Theory in $\mathbb{F}_q(t)$: Polynomials and function fields, Fermat's last theorem for polynomials, primes, unique factorization, the zeta function, the power reciprocity law and applications, abelian characters, Dirichlet's theorem on primes in arithmetic progressions

Week 5: Homework 1 due.

Weeks 5-8: Algebraic Function Fields: Algebraic function fields, valuations, Dedekind domains and their properties, algebraic curves and the dictionary between fields and curves, differentials and divisors, the Riemann-Roch theorem and its applications, zeta functions, the Weil conjectures for algebraic curves, elliptic curves and elliptic function fields.

Week 8: Homework 2 due.

Weeks 9-11: Extensions of Function Fields: Separable and inseparable extensions, primes and ramification, the Riemann-Hurwitz theorem, Lüroth's theorem, the *abc* conjecture for function fields and its applications, constant field extensions.

Week 12: Homework 3 due.

Weeks 12-14: Additional Topics: (some selection among these topics, according to student interest) S-units, L-functions, the Chebotarev density theorem, class number formulas, Artin's primitive root conjecture, cyclotomic function fields, the Brumer-Stark conjecture, the Riemann hypothesis for function fields.

Week 15: Homework 4 due.

Attendance Policy: It is expected that you will attend every class (see above under "Grades"), but I am willing to work with you if you will have to miss some of the lectures. Since this is a graduate-level topics course, it is very easy to get lost even if you only miss one day.

If you will be absent from a class activity due to a religious observance or practice, or for participation in a university-sanctioned event (e.g., university athletics), it is your responsibility to inform the instructor during the first week of class and provide appropriate documentation if required.

Statement on Academic Integrity: A commitment to the principles of academic integrity is essential to the mission of Northeastern University. Academic dishonesty violates the most fundamental values of an intellectual community and undermines the achievements of the entire University. Violations of academic integrity include (but are not limited to) cheating on assignments or exams, fabrication or misrepresentation of data or other work, plagiarism, unauthorized collaboration, and facilitation of others' dishonesty. Possible sanctions include (but are not limited to) warnings, grade penalties, course failure, suspension, and expulsion.

Statement on Accommodations: Any student with a disability is encouraged to meet with or otherwise contact the instructor during the first week of classes to discuss accommodations. The student must bring a current Memorandum of Accommodations from the Office of Student Disability Services.

Statement on Classroom Behavior: Disruptive classroom behavior will not be tolerated. In general, any behavior that impedes the ability of your fellow students to learn will be viewed as disruptive.

Statement on Inclusivity: Faculty are encouraged to address students by their preferred name and gender pronoun. If you would like to be addressed using a specific name or pronoun, please let your instructor know.

Statement on Evaluations: Students are requested to complete the TRACE evaluations at the end of the course.

Miscellaneous Disclaimer: The instructor reserves the right to change course policies, including the evaluation scheme of the course. Notice will be given in the event of any substantial changes.