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1. Identify each statement as either true or false, where the vector spaces U, V, W are finite-dimensional, the bases α, β, γ are ordered, and that S, T are linear transformations.

   (a) In $\mathbb{R}^3$, the orthogonal complement of the xy-plane is the z-axis.

   (b) If A is any $m \times n$ matrix, the orthogonal complement of the row space of A is the column space of A.

   (c) If W is a subspace of V where $\dim(W) = 4$ and $\dim(V) = 9$, then $\dim(W^\perp) = 5$.

   (d) If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is an orthonormal basis for V and $W = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$, then $W^\perp = \text{span}(\mathbf{e}_3, \mathbf{e}_4)$.

   (e) If $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis for V and $W = \text{span}(2\mathbf{e}_1 + \mathbf{e}_2)$, then $W^\perp = \text{span}(2\mathbf{e}_1 - \mathbf{e}_2)$.

   (f) If $\mathbf{v}$ is orthogonal to $\mathbf{w}_1$ and $\mathbf{w}_2$, and $W = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$, then $\mathbf{v}$ is in $W^\perp$.

   (g) The orthogonal projection of $\langle 1, 2, 3\rangle$ into the subspace spanned by $\langle 1, 1, 0\rangle$ is $\langle 2, 2, 0\rangle$.

   (h) If $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ where $\mathbf{w}$ is in W and $\mathbf{w}^\perp$ is in $W^\perp$, then $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2$.

   (i) The closest vector to $\langle 1, 2, 3\rangle$ inside the plane $x + y + z = 0$ in $\mathbb{R}^3$ is $\langle -1, 0, 1\rangle$.

   (j) If $A\mathbf{x} = \mathbf{c}$ is an inconsistent system of linear equations, then the best approximation of a solution is given by the solutions $\hat{\mathbf{x}}$ of $A^T\hat{\mathbf{x}} = A^T A\mathbf{c}$.

   (k) If $\mathbf{w}^\perp$ is a vector in $W^\perp$, then the orthogonal projection of $\mathbf{w}^\perp$ onto W is $\mathbf{w}^\perp$ itself.

   (l) If P is the standard matrix associated to an orthogonal projection, then $P^2 = P$.

   (m) If $T : V \to V$ and λ is a scalar, the set of vectors $\mathbf{v}$ with $T(\mathbf{v}) = \lambda\mathbf{v}$ is a subspace of V.

   (n) The eigenvalues of any matrix with real entries are always real numbers.

   (o) Any $n \times n$ matrix always has n distinct eigenvalues.

   (p) If the characteristic polynomial for A is $p(t) = t(t-1)^2$, then 0 is an eigenvalue of A.

   (q) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the 1-eigenspace of A has dimension 2.

   (r) If $\mathbf{v}_1$ is an eigenvector with eigenvalue λ_1 and $\mathbf{v}_2$ is an eigenvector with eigenvalue λ_2, then $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector with eigenvalue $\lambda_1 + \lambda_2$.

   (s) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors with distinct eigenvalues λ_1, λ_2, and λ_3, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

   (t) A linear map $T : V \to V$ is diagonalizable if and only if V has a basis of eigenvectors of T.

   (u) If A is invertible, then A is diagonalizable.

   (v) If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

   (w) If an $n \times n$ matrix has fewer than n distinct eigenvalues, then it is not diagonalizable.

   (x) If the characteristic polynomial for A is $p(t) = t^3 - t$, then $A^3 - A$ is the zero matrix.

   (y) Every real symmetric matrix has real eigenvalues and is diagonalizable.

   (z) If M is an $n \times n$ matrix with orthonormal columns, then $M^T M$ is the identity matrix.

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2. For each of the given subspaces W, find a basis for its orthogonal complement $W^\perp$:

   (a) $W = \text{span}[(1, 2, 3), (0, 1, 2)]$ inside $V = \mathbb{R}^3$ under the standard dot product.

   (b) $W = \text{span}[(1, 1, 1, 1), (1, 1, -1, -1)]$ inside $V = \mathbb{R}^4$ under the standard dot product.

   (c) $W =$ the plane $2x + y + 3z = 0$ inside $V = \mathbb{R}^3$ under the standard dot product.

   (d) $W = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$ inside $V = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of V.

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3. Find a basis for the subspace of $\mathbb{R}^4$ consisting of all vectors orthogonal to $\mathbf{v} = (1, 2, 3, 4)$.

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4. Let $\mathbf{w}_1 = (1, 0, 1, 2)$, $\mathbf{w}_2 = (0, 1, 2, -1)$, and $\mathbf{w}_3 = (-2, 1, 0, 1)$. Note that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are orthogonal.
   (a) Let $W_1 = \text{span}(\mathbf{w}_1)$. Find the orthogonal decomposition of the vector $\mathbf{v} = (6, 6, 6, 6)$ into the sum of a vector $\mathbf{w}$ in W_1 and a vector $\mathbf{w}^\perp$ orthogonal to W_1.
   (b) Find the orthogonal projection of the vector $\mathbf{v} = (6, 6, 6, 6)$ into the subspace $W_2 = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$ of $\mathbb{R}^4$.
   (c) Find the closest vector in the subspace $W_3 = \text{span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ to the vector $\mathbf{v} = (6, 6, 6, 6)$ in $\mathbb{R}^4$.
   (d) Find the standard matrix for orthogonal projection onto the subspace $W_4 = \text{span}(\mathbf{w}_1, \mathbf{w}_3)$ of $\mathbb{R}^4$.
   (e) Find a basis for the orthogonal complement of the subspace $W_5 = \text{span}(\mathbf{w}_2, \mathbf{w}_3)$ of $\mathbb{R}^4$.

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5. Use least squares to solve each of the following problems:
   (a) Find the least squares solution $\hat{\mathbf{x}}$ to the inconsistent system $\begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.
   (b) Find the vector $\mathbf{x} = (x, y)$ that is closest to a solution to the system $x + y = 1$, $2x + y = 2$, $x + 2y = 3$.
   (c) Find the line $y = mx + b$ of best fit to the data set $\{(2, 2), (-1, -4), (-2, -6)\}$.
   (d) Find the line $y = mx + b$ of best fit to the data set $\{(0, 1), (1, 2), (2, 4), (3, 4)\}$.
   (e) Find the parabola $y = ax^2 + bx + c$ of best fit to the data set $\{(-2, 1), (-1, 0), (1, 2), (2, 4)\}$.

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6. Find the matrix of orthogonal projection onto each subspace of $\mathbb{R}^n$ (with respect to the standard basis):
   (a) The subspace of $\mathbb{R}^3$ spanned by the vector $(1, -2, 3)$.
   (b) The subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 1, 2)$ and $(2, 0, 2, 1)$.
   (c) The plane $x + 2y - z = 0$ in $\mathbb{R}^3$.

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7. For each matrix A, find (i) the characteristic polynomial and eigenvalues of A, (ii) a basis for each eigenspace of A, and (iii) whether A is diagonalizable and if so find an invertible matrix Q with $D = Q^{-1}AQ$ diagonal:
(a) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -2 & 4 \\ -3 & 2 & -6 \\ -2 & 2 & -5 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{bmatrix}$

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8. Suppose the characteristic polynomial of the matrix A is $p(t) = t^4(t - 2)^2(t + 4)$.
   (a) Find the dimensions of A.
   (b) Find the eigenvalues of A and their multiplicities.
   (c) Find the determinant and the trace of A.
   (d) Find all possible values for each of the dimensions of the eigenspaces of A. Under what condition(s) on those dimensions will A be diagonalizable?
   (e) If A is diagonalizable, find a possible diagonalization.

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9. For each symmetric matrix A, find an orthogonal diagonalization (i.e., an orthogonal matrix Q such that $D = Q^{-1}AQ$ is diagonal):
(a) $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

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10. Suppose A is a 3×3 matrix with eigenvectors $\mathbf{v}_1$ of eigenvalue 2, $\mathbf{v}_2$ of eigenvalue 3, and $\mathbf{v}_3$ of eigenvalue 5.
   (a) In terms of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, find $A\mathbf{v}_1$, $A(\mathbf{v}_2 + \mathbf{v}_3)$, and $A(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$.
   (b) Explain why A is diagonalizable and find a diagonalization.
   (c) If $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (2, 4, 3)$, and $\mathbf{v}_3 = (2, 1, 1)$, find a matrix product formula for A (you do not have to evaluate it).
